Diophantine quadruples and Fibonacci numbers

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Abstract

A Diophantine m-tuple is a set of m positive integers with the property that product of any two of its distinct elements is one less then a square. In this survey we describe some problems and results concerning Diophantine m-tuples and their connections with Fibonacci numbers.

1 Introduction

The Greek mathematician Diophantus of Alexandria found four positive rationals 1/16, 33/16, 17/4, 105/16 with the property that the product of any two of them increased by 1 is a square of a rational number (see [9, 10]). The first set of four positive integers with the same property, the set $\{1, 3, 8, 120\}$, was found by Fermat. Indeed, we have

$$1 \cdot 3 + 1 = 2^2, \quad 1 \cdot 8 + 1 = 3^2, \quad 1 \cdot 120 + 1 = 11^2,$$

$$3 \cdot 8 + 1 = 5^2, \quad 3 \cdot 120 + 1 = 19^2, \quad 8 \cdot 120 + 1 = 31^2.$$

Definition 1 A set of m positive integers $\{a_1, a_2, \ldots, a_m\}$ is called *a Diophantine m-tuple* if $a_i a_j + 1$ is a perfect square for all $1 \le i < j \le m$. If $a_1, a_2, \ldots, a_m$ are nonzero rationals with the same property, then such set is called *a rational Diophantine m-tuple.*

In 1969, Baker and Davenport [1] proved that the Fermat's set cannot be extended to a Diophantine quintuple. In 1998, Dujella and Pethő proved that even the Diophantine pair $\{1, 3\}$ cannot be extended to a Diophantine quintuple. A "folklore" conjecture is that *there does not exist a Diophantine quintuple.* However, the first absolute upper bound for the size of Diophantine tuples was given in 2001, when we proved that there does not exist a Diophantine 9-tuple (see [20]). This result was recently improved in [21], where we proved that there does not exist a Diophantine sextuple and that there are only finitely many, effectively computable, Diophantine quintuples.

On the other hand, no absolute upper bound for the size of rational Diophantine tuples is known. The first rational Diophantine quintuple, the set

$$\{1,\ 3,\ 8,\ 120,\ 777480/8288641\},$$

was found by Euler. In 1999, Gibbs found several examples of rational Diophantine sextuples, e.g.

$$\{11/192,\ 35/192,\ 155/27,\ 512/27,\ 1235/48,\ 180873/16\},$$

$$\{17/448,\ 265/448,\ 2145/448,\ 252,\ 23460/7,\ 2352/7921\}.$$

In the present paper, we will describe some connections of Diophantine m-tuples and Fibonacci numbers. There are several reasons why Fibonacci numbers play an important role in this area. The starting point in the construction of larger set with the property of Diophantus is usually some identity of the form $ab+1=r^2$. The obvious one

$$(k-1)(k+1)+1=k^2,$$

leads to an infinite family of Diophantine quadruples

$$\{k-1,\ k+1,\ 4k,\ 16k^3-4k\}. \tag{1}$$

Concerning this family, we proved in [15] that if $k \geq 2$ is an integer and if $\{k-1, k+1, 4k, d\}$ is a Diophantine quadruple, then d has to be $16k^3-4k$. Another popular starting identity is well-known Cassini's identity. One of its equivalent formulations is

$$F_k \cdot F_{k+2} + (-1)^k = F_{k+1}^2,$$

and this is the basis for the construction of so called Hoggatt-Bergum's quadruple, which will be discussed in Section 2.

In Section 4, we will see that Fibonacci numbers satisfy some, more involved, pairs of identities which fit very nice in the general theory of Diophantine quadruples.

There is a very simple way how one can extend given Diophantine pair $\{a,b\}$ to a Diophantine triple. Namely, if $ab+1=r^2$, then $\{a,b,a+b+2r\}$ is the Diophantine triple. If we iterate this construction, we obtain the sequence with the property that its every three successive elements form a Diophantine triple. The k-th element of this sequence is

$$F_{k-2}^2 a + F_{k-1}^2 b + 2F_{k-2}F_{k-1} r$$

(see [25]).

2 Hoggatt-Bergum's quadruple

In 1977, Hoggatt and Bergum [27] proved that for any positive integer k, the set

$$\{F_{2k},\, F_{2k+2},\, F_{2k+4},\, 4F_{2k+1}F_{2k+2}F_{2k+3}\} \tag{2}$$

is a Diophantine quadruple. Indeed,

$$\begin{aligned}
F_{2k}\cdot F_{2k+2}+1 &= F_{2k+1}^2,\\
F_{2k}\cdot F_{2k+4}+1 &= F_{2k+2}^2,\\
F_{2k}\cdot 4F_{2k+1}F_{2k+2}F_{2k+3}+1 &= (2F_{2k+1}F_{2k+2}-1)^2,\\
F_{2k+2}\cdot F_{2k+4}+1 &= F_{2k+3}^2,\\
F_{2k+2}\cdot 4F_{2k+1}F_{2k+2}F_{2k+3}+1 &= (2F_{2k+2}^2+1)^2,\\
F_{2k+4}\cdot 4F_{2k+1}F_{2k+2}F_{2k+3}+1 &= (2F_{2k+2}F_{2k+3}+1)^2.
\end{aligned}$$

They also conjectured that the fourth element of the set (2) is unique. In 1999, we were able to prove this conjecture ([17]).

Theorem 1 *Let k be a positive integer. If the set $\{F_{2k}, F_{2k+2}, F_{2k+4}, d\}$ is a Diophantine quadruple, then $d = 4F_{2k+1}F_{2k+2}F_{2k+3}$.*

We outline the proof of Theorem 1. The proof of the result, mentioned in Introduction, that there exist only finitely many Diophantine quintuples, although much more involved, uses the similar ideas.

Assume that $\{a, b, c\}$ is a Diophantine triple, and we want to find all extensions of this triple to a Diophantine quadruple $\{a, b, c, d\}$. Eliminating d from

$$ad + 1 = x^2, \quad bd + 1 = y^2, \quad cd + 1 = z^2,$$

we obtain the following system of Pellian equations

$$\begin{aligned}
ay^2 - bx^2 &= a - b,\\
az^2 - cx^2 &= a - c.
\end{aligned}$$

All solutions of a single Pellian equation are contained in the union of finitely many binary recursive sequences. Therefore, our problem reduces to finitely many equations of the form $v_m = w_n$, where $\{v_m\}$ and $\{w_n\}$ are binary recursive sequences.

In the case $\{a, b, c\} = \{F_{2k}, F_{2k+2}, F_{2k+4}\}$, it can be shown that we have, essentially, only one such equation. Namely, in this case, we have to find an intersection of two two-sided sequences, defined by

$$v_0 = 1,\ \ v_1 = F_{2k+2},\ \ v_{m+2} = 2F_{2k+1}v_{m+1} - v_m,\ \ m \in \mathbb{Z},$$
$$w_0 = 1,\ \ w_1 = F_{2k} + F_{2k+2},\ \ w_{n+2} = 2F_{2k+2}w_{n+1} - w_n,\ \ n \in \mathbb{Z}.$$

We claim that the only solutions of the equation $v_m = w_n$, $m, m \in \mathbb{Z}$ are $v_0 = w_0 = 1$ and $v_2 = w_{-2} = 2F_{2k+1}F_{2k+2} - 1$. These solutions correspond to $d = 0$ and $d = 4F_{2k+1}F_{2k+2}F_{2k+3}$.

In the proof of this statement we derive a lower bound and an upper bound for solutions. These bounds have the property that the lower bound is greater that the upper bound for sufficiently large k. These bounds are obtained by two different transformations of the original exponential equation $v_m = w_n$.

We start with trivial observation that the equality $v_m = w_n$ implies the congruence $v_m \equiv w_n \pmod{N}$ for all positive integers N. We would like to choose the modulus N in such way that: **1)** $\{v_m \bmod N\}$ is a polynomial (not exponential) sequence in m; **2)** $\{w_n \bmod N\}$ has small period. The both conditions are satisfied by $N = 2F_{2k}F_{2k+2}$. Namely, we have

$$\begin{aligned} v_{2m} &\equiv 2mF_{2k+2}^2 - (2m-1) \pmod{2F_{2k}F_{2k+2}}, \\ v_{2m+1} &\equiv F_{2k+2} + 2(m-1)F_{2k} \pmod{2F_{2k}F_{2k+2}}, \end{aligned}$$

$$w_{4n} \equiv 1 \pmod{2F_{2k}F_{2k+2}}, \quad w_{4n+1} \equiv F_{2k} + F_{2k+2} \pmod{2F_{2k}F_{2k+2}},$$
$$w_{4k+2} \equiv 2F_{2k+2}^2 - 1 \pmod{2F_{2k}F_{2k+2}}, \quad w_{4k+3} \equiv F_{2k+1} \pmod{2F_{2k}F_{2k+2}}.$$

It is easy to see that these congruences imply that from $v_m = w_n$ it follows $m \equiv 0$ or $2 \pmod{2F_{2k+2}}$. This gives very good lower bound for solutions. Indeed, if $d \neq 4F_{2k+1}F_{2k+2}F_{2k+3}$, then $m \neq 2$, and therefore

$$m \geq 2F_{2k+2}. \tag{3}$$

An upper bound can be obtained using Baker's theory. Namely, we can transform our exponential equation $v_m = w_n$ into a logarithmic inequality, and then we may apply results on linear forms in logarithms of algebraic numbers. In [17], we used a theorem of Baker and Wüstholz [2]. We obtained the following upper bound

$$\frac{m}{\log m} < 6.423 \cdot 10^{15} \log^2 F_{2k},$$

which clearly contradicts (3) for large k. Indeed, this finishes the proof for $k \geq 49$. In remaining 48 cases we solved the corresponding systems of Pellian equations by a version of Baker-Davenport reduction procedure ([1, 22]). Note that the case $k = 1$ is exactly the original result of Baker and Davenport. ■

As a simple consequence of Theorem 1 we obtain the result that if $\{F_{2k}, F_{2k+2}, F_{2k+4}, d\}$ is a Diophantine quadruple, then d cannot be a Fibonacci number. This answers the question posed by Jones [30]. The statement follows directly from Theorem 1 and the relation

$$F_{6k+5} < 4F_{2k+1}F_{2k+2}F_{2k+3} < F_{6k+6},$$

proved by Jones.

There are many papers devoted to various generalizations of Hoggatt-Bergum's quadruple. Let us briefly mention some of them.

In [33], Morgado showed that the product of any two distinct elements of the set

$$\{F_k, F_{k+2r}, F_{k+4r}, 4F_{k+r}F_{k+2r}F_{k+3r}\}$$

increased by $F_a^2 F_b^2$ or $-F_a^2 F_b^2$, for suitable positive integers a and b, is a perfect square. The similar results are valid for more general binary recursive sequences (see [28, 35, 36, 38, 39, 40]).

In [14], the following direct generalization of the Hoggatt-Bergum's result was obtained. Let the sequences $u_k = u_k(p)$ and $g_k = g_k(p)$ be defined by

$$u_0 = 0, \ \ u_1 = 1, \ \ u_k = pu_{k-1} + u_{k-2}, \ \ k \geq 2;$$

$$g_0 = 0, \ \ g_1 = 1, \ \ g_k = pg_{k-1} - g_{k-2}, \ \ k \geq 2,$$

where p is an integer. Then

$$\{u_{2k},\ u_{2k+2},\ 2u_{2k} + (p+2)u_{2k+1},\ 4u_{2k+1}((p+2)u_{2k+1}^2 + 2u_{2k}u_{2k+1} + 1)\} \tag{4}$$

and

$$\{g_k,\ g_{k+2},\ (p-2)g_{k+1},\ 4g_{k+1}((p-2)g_{k+1}^2 - 1)\} \tag{5}$$

are $D(1)$-quadruples. Since $u_k(1) = F_k$ and $g_k(3) = F_{2k}$, these quadruples are generalizations of Hoggatt-Bergum's quadruple. Moreover, since $g_k(2) = k$, we may consider (5) as a common generalization of Hoggatt-Bergum's quadruple (2) and the polynomial quadruple (1).

If we concentrate our attention to the first three elements of Hoggatt-Bergum's quadruple, we see that the sequence of Fibonacci numbers with even subscripts $\{F_{2k}\}$ has one remarkable property. If we choose three successive elements of this sequence, then the product of any two of them increased by 1 is a perfect square. In [8], Deshpande and Dujella characterized all nondegenerate binary recursive sequences $\{G_k\}$ of the form $G_k = AG_{k-1} - G_{k-2}$, which possess the property that there exist an integer n such that $G_kG_{k+1} + n$ and $G_kG_{k+2} + n$ are perfect squares for all $k \geq 0$. The result is that this assumption implies $A = 3$ and $n = G_0^2 - 3G_0G_1 + G_1^2$. In the other words, $G_k = G_1F_{2k} - G_0F_{2k-2}$. Therefore, we see that this property is very closely related with Fibonacci numbers of even subscripts.

Three successive elements of some other binary recursive sequences also posses some interesting properties related with the property of Diophantus (see [3, 6, 7, 31, 37]).

3 A family of elliptic curves

In the previous section, we have shown that all integer solutions of the system

$$F_{2k}d + 1 = \square, \quad F_{2k+2}d + 1 = \square, \quad F_{2k+4}d + 1 = \square, \tag{6}$$

are given by $d = 0$ and $d = 4F_{2k+1}F_{2k+2}F_{2k+3}$. Multiplying three conditions from (6), we obtain the elliptic curve

$$E_k: \quad y^2 = (F_{2k}x + 1)(F_{2k+2}x + 1)(F_{2k+4}x + 1). \tag{7}$$

Since we can solve the system (6) completely, we may try to find all integer points on the elliptic curve (7). In [19], we were able to do this, under assumption that the rank is "smallest possible".

Theorem 2 *Let $k \geq 2$ be an integer. If* $\operatorname{rank} E_k(\mathbb{Q}) = 1$, *then all integer points on E_k are given by*

$$\begin{aligned}(x, y) \in \{(0, \pm 1),\ (4F_{2k+1}F_{2k+2}F_{2k+3}, \pm(2F_{2k+1}F_{2k+2} - 1) \\ \times (2F_{2k+2}^2 + 1)(2F_{2k+2}F_{2k+3} + 1)\}.\end{aligned} \tag{8}$$

In the following table we list the values of $r_k = \operatorname{rank}(E_k(\mathbb{Q}))$ which we were able to compute (unconditionally) using Cremona's program MWRANK [5]:

k	1	2	3	4	5	6	7	8	9	12	13	14	16	17	20	23	25	29	31
r_k	1	1	2	2	3	1	3	2	3	1	3	2	1	2	1	2	2	2	2

We also proved that for $2 \leq k \leq 50$, all integer points on E_k are given by (8). For $k = 1$ there is one additional integer point, $(-1, 0)$.

Results analogous to Theorem 7 for the families of elliptic curves corresponding to the Diophantine triples of the form $\{k-1, k+1, 4k\}$ and $\{1, 3, c\}$, were proved in [18] and [23] (see also [29]).

4 Diophantine quadruples for squares of Fibonacci numbers

Definition 2 Let n be an integer. A set of positive integers $\{a_1, a_2, \ldots, a_m\}$ is called *a $D(n)$-m-tuple* (or *a Diophantine m-tuple with the property $D(n)$*, or *a P_n-set of size m*) if $a_i a_j + n$ is a perfect square for all $1 \leq i < j \leq m$.

Several authors considered the problem for which integers n there exists a $D(n)$-quadruple. The first part of the answer was given in 1985 by Brown [4], Gupta & Singh [25] and Mohanty & Ramasamy [32]. They proved, independently, that if $n \equiv 2 \pmod 4$, then there does not exist a $D(n)$-quadruple. In 1993, we gave the second part of the answer by proving that if $n \not\equiv 2 \pmod 4$ and $n \notin \{-4, -3, -1, 3, 5, 8, 12, 20\}$, then there exist at least one $D(n)$-quadruple.

Questions of different types appear when n is a perfect square. The sets with the property $D(l^2)$, where l is a positive integer, were systematically studied in [12]. It was proved that for any $D(l^2)$-pair $\{a, b\}$, such that ab is not a perfect square, there exist infinitely $D(l^2)$-quadruples of the form $\{a, b, c, d\}$. This is the generalization of well known fact for $l = 1$ (see [26]). The proof of this result is based on the construction of three double sequences $x_{n,m}$, $y_{n,m}$ and $z_{n,m}$ which are defined as follows. Let $ab + l^2 = r^2$ and let s and t be the minimal solutions the Pell equation $S^2 - abT^2 = 1$. Define

$$y_{0,0} = l, \quad z_{0,0} = l, \quad y_{1,0} = r + a, \quad z_{1,0} = r + b,$$
$$y_{n+1,0} = \tfrac{2r}{l} y_{n,0} - y_{n-1,0}, \quad z_{n+1,0} = \tfrac{2r}{l} z_{n,0} - z_{n-1,0}, \quad n \in \mathbb{Z},$$
$$y_{n,1} = s y_{n,0} + at z_{n,0}, \quad z_{n,1} = bt y_{n,0} + s z_{n,0}, \quad n \in \mathbb{Z},$$
$$y_{n,m+1} = 2s y_{n,m} - y_{n,m-1}, \quad z_{n,m+1} = 2s z_{n,m} - z_{n,m-1}, \quad n, m \in \mathbb{Z}.$$

The desired quadruples have the form $\{a, b, x_{n,m}, x_{n+1,m}\}$, where

$$x_{n,m} = (y_{n,m}^2 - l^2)/a = (z_{n,m}^2 - l^2)/b\,.$$

In [12] and [16], the following result was proved.

Theorem 3 *The sets* $\{a, b, x_{0,m}, x_{-1,m}\}$*,* $m \notin \{-1, 0, 1\}$*, and* $\{a, b, x_{0,m}, x_{1,m}\}$*,* $m \notin \{-2, -1, 0\}$*, are Diophantine quadruples with the property* $D(l^2)$*.*

If we have a pair of identities of the form

$$ab + l^2 = r^2 \quad \text{and} \quad s^2 - abt^2 = 1,$$

then we can construct the sequence $x_{m,n}$ and, by Theorem 3, obtain infinitely many Diophantine quadruples with the property $D(l^2)$. There are several pairs of identities for Fibonacci and Lucas numbers, which have the above form. For example, starting with the identities

$$\begin{aligned} 4F_{k-1}F_{k+1} + F_k^2 &= L_k^2, \\ 4F_{k-1}F_k^2F_{k+1} + 1 &= (F_k^2 + F_{k-1}F_{k+1})^2, \end{aligned}$$

we obtain, e.g. the $D(F_k^2)$-quadruples

$$\{2F_{k-1},\, 2F_{k+1},\, 2F_k^3F_{k+1}F_{k+2},\, 2F_{k+1}F_{k+2}F_{k+3}(2F_{k+1}^2 - F_k^2)\},$$

$$\{F_{k-1},\, 4F_{k+1},\, F_{k-2}F_{k-1}F_{k+1}(2F_k^2 - F_{k-1}^2),\, F_k^3F_{k+2}F_{k+3}\},$$

$$\{4F_{k-1},\, F_{k+1},\, F_{k-2}F_{2k-2}F_{2k-1},\, F_k^3L_kL_{k+1}\}$$

(see [13]).

In general, $x_{2,0}$ need not to be an integer. Indeed, $x_{2,0} = \frac{4r(r+a)(r+b)}{l^2}$. But if $x_{2,0}$ is a positive integer, then $\{a, b, x_{1,0}, x_{2,0}\}$ is a $D(l^2)$-quadruple. With this method, some formulas for quadruples with the properties $D(1)$, $D(4)$, $D(9)$ and $D(64)$, in terms of Fibonacci and Lucas numbers, were obtained in [11, 12]. This idea was also applied in [13] to the Morgado identity [34]:

$$F_{k-3}F_{k-2}F_{k-1}F_{k+1}F_{k+2}F_{k+3} + L_k^2 = \Big(F_k(2F_{k-1}F_{k+1} - F_k^2)\Big)^2.$$

Among the others, the following $D(L_k^2)$-quadruple was obtained:

$$\{F_{k-3}F_{k-2}F_{k+1},\, F_{k-1}F_{k+2}F_{k+3},\, F_kL_k^2,\, 4F_{k-1}^2F_kF_{k+1}^2(2F_{k-1}F_{k+1} - F_k^2)\}.$$

In [14], all results from this section were generalized to the sequences $\{u_k\}$ and $\{g_k\}$, defined in Section 2.

References

[1] A. Baker and H. Davenport, *The equations* $3x^2 - 2 = y^2$ *and* $8x^2 - 7 = z^2$, Quart. J. Math. Oxford Ser. (2) **20** (1969), 129–137.

[2] A. Baker and G. Wüstholz, *Logarithmic forms and group varieties*, J. Reine Angew. Math. **442** (1993), 19–62.

[3] A. F. Beardon and M. N. Deshpande, *Diophantine triples*, The Mathematical Gazette **86** (2002), 258–260.

[4] E. Brown, *Sets in which* $xy + k$ *is always a square*, Math. Comp. **45** (1985), 613–620.

[5] J. E. Cremona, Algorithms for Modular Elliptic Curves, Cambridge Univ. Press, 1997.

[6] M. N. Deshpande, *One interesting family of diophantine triplets*, Internat. J. Math. Ed. Sci. Tech. **33** (2002), 253–256.

[7] M. N. Deshpande and E. Brown, *Diophantine triplets and the Pell sequence*, Fibonacci Quart. **39** (2001), 242–249.

[8] M. N. Deshpande and A. Dujella, *An interesting property of a reccurence related to the Fibonacci sequence*, Fibonacci Quart. **40** (2002), 157–160.

[9] L. E. Dickson, History of the Theory of Numbers, Vol. 2, Chelsea, New York, 1966, pp. 513–520.

[10] Diophantus of Alexandria, Arithmetics and the Book of Polygonal Numbers, (I. G. Bashmakova, Ed.), Nauka, Moscow, 1974 (in Russian), pp. 103–104, 232.

[11] A. Dujella, *A problem of Diophantus and Fibonacci numbers*, Matematika (Zagreb) **19**(3) (1990), 45–52 (in Croatian).

[12] A. Dujella, *Generalization of a problem of Diophantus*, Acta Arith. **65** (1993), 15–27.

[13] A. Dujella, *Diophantine quadruples for squares of Fibonacci and Lucas numbers*, Portugaliae Math. **52** (1995), 305–318.

[14] A. Dujella, *Generalized Fibonacci numbers and the problem of Diophantus*, Fibonacci Quart. **34** (1996), 164–175.

[15] A. Dujella, *The problem of the extension of a parametric family of Diophantine triples*, Publ. Math. Debrecen **51** (1997), 311–322.

[16] A. Dujella, *A problem of Diophantus and Pell numbers*, Application of Fibonacci Numbers, Vol. 7 (G. E. Bergum, A. N. Philippou, A. F. Horadam, eds.), Kluwer, Dordrecht, 1998, pp. 61–68.

[17] A. Dujella, *A proof of the Hoggatt-Bergum conjecture*, Proc. Amer. Math. Soc. **127** (1999), 1999-2005.

[18] A. Dujella, *A parametric family of elliptic curves*, Acta Arith. **94** (2000), 87–101.

[19] A. Dujella, *Diophantine m-tuples and elliptic curves*, J. Theor. Nombres Bordeaux **13** (2001), 111–124.

[20] A. Dujella, *An absolute bound for the size of Diophantine m-tuples*, J. Number Theory **89** (2001), 126–150.

[21] A. Dujella, *There are only finitely many Diophantine quintuples*, preprint.

[22] A. Dujella and A. Pethő, *A generalization of a theorem of Baker and Davenport*, Quart. J. Math. Oxford Ser. (2) **49** (1998), 291–306.

[23] A. Dujella and A. Pethő, *Integer points on a family of elliptic curves*, Publ. Math. Debrecen **56** (2000), 321–335.

[24] P. Gibbs, *Some rational Diophantine sextuples*, preprint, `math.NT/9902081`.

[25] H. Gupta and K. Singh, *On k-triad sequences*, Internat. J. Math. Math. Sci. **5** (1985), 799–804.

[26] P. Heichelheim, *The study of positive integers (a, b) such that $ab+1$ is a square*, The Fibonacci Quarterly **17** (1979), 269–274.

[27] V. E. Hoggatt and G. E. Bergum, *A problem of Fermat and the Fibonacci sequence*, Fibonacci Quart. **15** (1977), 323–330.

[28] A. F. Horadam, *Generalization of a result of Morgado*, Portugaliae Math. **44** (1987), 131–136.

[29] M. J. Jacobson, Jr. and H. C. Williams, *Modular arithmetic on elements of small norm in quadratic fields*, Des. Codes and Cryptogr. **27** (2002), 93–110.

[30] B. W. Jones, *A second variation on a problem of Diophantus and Davenport*, Fibonacci Quart. **16** (1978), 155–165.

[31] C. Long and G. E. Bergum, *On a problem of Diophantus*, Application of Fibonacci Numbers, Vol 2 (A. N. Philippou, A. F. Horadam, G. E. Bergum, eds.), Kluwer, Dordrecht, 1988, pp. 183–191.

[32] S. P. Mohanty and A. M. S. Ramasamy, *On $P_{r,k}$ sequences*, Fibonacci Quart. **23** (1985), 36–44.

[33] J. Morgado, *Generalization of a result of Hoggatt and Bergum on Fibonacci numbers*, Portugaliae Math. **42** (1983–1984), 441–445.

[34] J. Morgado, *Note on some results of A. F. Horadam and A. G. Shannon concerning a Catalan's identity on Fibonacci numbers*, Portugaliae Mathematica, **44** (1987), 243–252.

[35] J. Morgado, *Note on a Shannon's theorem concerning the Fibonacci numbers and Diophantine quadruples*, Portugaliae Math. **48** (1991), 429–439.

[36] J. Morgado, *Note on the Chebyshev polynomials and applications to the Fibonacci numbers*, Portugaliae Math. **52** (1995), 363–378.

[37] A. M. S. Ramasamy, *A remarkable sequence*, Banyan Mathematical Journal **2** (1995), 69–76.

[38] A. G. Shannon, *Fibonacci numbers and Diophantine quadruples: Generalizations of results of Morgado and Horadam*, Portugaliae Math. **45** (1988), 165–169.

[39] Gh. Udrea, *A note on the sequence* (W_n) *of A. F. Horadam*, Portugaliae Math. **53** (1996), 143–155.

[40] Gh. Udrea, *A problem of Diophantus-Fermat and Chebyshev polynomials of the first kind*, Rev. Roumaine Math. Pures Appl. **45** (2000), 531–535.