Newton's formula and continued fraction expansion of $\sqrt{d}$

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Abstract

It is known that if the period $s(d)$ of the continued fraction expansion of $\sqrt{d}$ satisfies $s(d) \leq 2$, then all Newton's approximants $R_n = \frac{1}{2}(\frac{p_n}{q_n} + \frac{dq_n}{p_n})$ are convergents of $\sqrt{d}$, and moreover we have $R_n = \frac{p_{2n+1}}{q_{2n+1}}$ for all $n \geq 0$. Motivated with this fact we define two numbers $j = j(d,n)$ and $b = b(d)$ by $R_n = \frac{p_{2n+1+2j}}{q_{2n+1+2j}}$ if R_n is a convergent of $\sqrt{d}$; $b = |\{n : 0 \leq n \leq s-1 \text{ and } R_n \text{ is a convergent of } \sqrt{d}\}|$. The question is how large the quantities $|j|$ and b can be. We prove that $|j|$ is unbounded and give some examples which support a conjecture that b is unbounded too. We also discuss the magnitude of $|j|$ and b compared with d and $s(d)$.

1 Introduction

Let d be a positive integer which is not a perfect square. The simple continued fraction expansion of $\sqrt{d}$ has the form

$$\sqrt{d} = [a_0; \overline{a_1, a_2, \ldots, a_{s-1}, 2a_0}].$$

Here $s = s(d)$ denotes the length of the shortest period in the expansion of $\sqrt{d}$. Moreover, the sequence $a_1, \ldots, a_{s-1}$ is symmetrical, i.e. $a_i = a_{s-i}$ for $i = 1, \ldots, s-1$.

This expansion can be obtained using the following algorithm:

$$\begin{gathered} a_0 = \lfloor \sqrt{d} \rfloor, \quad b_1 = a_0, \quad c_1 = d - a_0^2, \\ a_{n-1} = \left\lfloor \frac{a_0 + b_{n-1}}{c_{n-1}} \right\rfloor, \quad b_n = a_{n-1}c_{n-1} - b_{n-1}, \quad c_n = \frac{d - b_n^2}{c_{n-1}} \quad \text{for } n \geq 2 \end{gathered} \tag{1}$$

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(see [Sierpiński 1987, p. 319]).

Let $\frac{p_n}{q_n}$ be the nth convergent of $\sqrt{d}$. Then

$$\frac{1}{(a_{n+1}+2)q_n^2} < |\sqrt{d} - \frac{p_n}{q_n}| < \frac{1}{a_{n+1}q_n^2} \tag{2}$$

(see [Schmidt 1980, p. 23]). Furthermore, if there is a rational number $\frac{p}{q}$ with $q \geq 1$ such that

$$|\sqrt{d} - \frac{p}{q}| < \frac{1}{2q^2}\,, \tag{3}$$

then $\frac{p}{q}$ equals one of the convergents of $\sqrt{d}$.

Another method for the approximation of $\sqrt{d}$ is by Newton's formula

$$x_{k+1} = \frac{1}{2}\Big(x_k + \frac{d}{x_k}\Big)\,. \tag{4}$$

In this paper we will discuss connections between these two methods. More precisely, if $\frac{p_n}{q_n}$ is a convergent of $\sqrt{d}$, the questions is whether

$$R_n = \frac{1}{2}\Big(\frac{p_n}{q_n} + \frac{dq_n}{p_n}\Big)$$

is also a convergent of $\sqrt{d}$.

This question was discussed by several authors. It was proved by Mikusiński [1954] (see also [Clemens at al. 1995; Elezović 1997; Sharma 1959]) that

$$R_{ks-1} = \frac{p_{2ks-1}}{q_{2ks-1}},$$

and if $s = 2t$ then

$$R_{kt-1} = \frac{p_{2kt-1}}{q_{2kt-1}}$$

for all positive integers k. These results imply that if $s(d) = 1$ or 2, then all approximants R_n are convergents of $\sqrt{d}$. Moreover, under these assumptions we have

$$R_n = \frac{p_{2n+1}}{q_{2n+1}} \tag{5}$$

for all $n \geq 0$.

2 Which convergents may appear?

Lemma 1

$$R_n - \sqrt{d} = \frac{q_n}{2p_n}\Big(\frac{p_n}{q_n} - \sqrt{d}\Big)^2$$

PROOF.

$$\begin{aligned} 2(R_n - \sqrt{d}) &= (\frac{p_n}{q_n} - \sqrt{d}) + (\frac{dq_n}{p_n} - \sqrt{d}) = (\frac{p_n}{q_n} - \sqrt{d}) - \frac{\sqrt{d}q_n}{p_n}(\frac{p_n}{q_n} - \sqrt{d}) \\ &= \frac{q_n}{p_n}(\frac{p_n}{q_n} - \sqrt{d})^2 \end{aligned}$$

∎

Theorem 1 *If $R_n = \frac{p_k}{q_k}$, then k is odd.*

PROOF. Since $\frac{p_l}{q_l} > \sqrt{d}$ if and only if l is odd, and by Lemma 1 we have $R_n > \sqrt{d}$, we conclude that k is odd. ∎

Assume that R_n is a convergent of $\sqrt{d}$. Then by Theorem 1 we have

$$R_n = \frac{p_{2n+1+2j}}{q_{2n+1+2j}}$$

for an integer $j = j(d, n)$. We have already seen that if $s(d) \leq 2$ then $j(d, n) = 0$. In [Elezović 1997; Komatsu 1999; Mikusiński 1954] some examples can be found with $j = \pm 1$. We would like to investigate the problem how large $|j|$ can be.

The following result of Komatsu [1999] shows that all periods of the continued fraction expansions of $\sqrt{d}$ have the same behavior concerning the questions in which we are interested, i.e. we may concentrate our attention on R_i for $0 \leq i \leq s - 1$.

Lemma 2 (Komatsu 1999) *For $n = 0, 1, \ldots, \lfloor s/2 \rfloor$ there exist α_n such that*

$$\begin{aligned} R_{ks+n-1} &= \frac{\alpha_n p_{2ks+2n} + p_{2ks+2n-1}}{\alpha_n q_{2ks+2n} + q_{2ks+2n-1}} \quad \textit{for all } k \geq 0, \textit{ and} \\ R_{ks-n-1} &= \frac{p_{2ks-2n-1} - \alpha_n p_{2ks-2n-2}}{q_{2ks-2n-1} - \alpha_n q_{2ks-2n-2}} \quad \textit{for all } k \geq 1. \end{aligned}$$

The following lemma reduces further our problem to the half-periods.

Lemma 3 *Let* $0 \le n \le s/2$. *If* $R_n = \frac{p_{2n+1+2j}}{q_{2n+1+2j}}$, *then*

$$R_{s-n-2} = \frac{p_{2(s-n-2)+1-2j}}{q_{2(s-n-2)+1-2j}}.$$

PROOF. If

$$\begin{aligned}\begin{pmatrix} p_{2n+1+2j} & q_{2n+1+2j} \\ p_{2n+2j} & q_{2n+2j} \end{pmatrix} &= \begin{pmatrix} a_{2n+1+2j} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{2n+3} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_{2n+2} & q_{2n+2} \\ p_{2n+1} & q_{2n+1} \end{pmatrix} \\ &= \begin{pmatrix} d & c \\ f & e \end{pmatrix} \begin{pmatrix} p_{2n+2} & q_{2n+2} \\ p_{2n+1} & q_{2n+1} \end{pmatrix}, \qquad (6)\end{aligned}$$

then

$$\begin{pmatrix} p_{2s-2n-2-2j} & q_{2s-2n-2-2j} \\ p_{2s-2n-3-2j} & q_{2s-2n-3-2j} \end{pmatrix} = \begin{pmatrix} -e & f \\ c & -d \end{pmatrix} \begin{pmatrix} p_{2s-2n-3} & q_{2s-2n-3} \\ p_{2s-2n-4} & q_{2s-2n-4} \end{pmatrix}. \quad (7)$$

By the assumption and formula (6), we have

$$R_n = \frac{p_{2n+1+2j}}{q_{2n+1+2j}} = \frac{p_{2n+1} + \frac{d}{c} p_{2n+2}}{q_{2n+1} + \frac{d}{c} q_{2n+2}}.$$

Now Lemma 2 and formula (7) imply

$$R_{s-n-2} = \frac{p_{2s-2n-3} - \frac{d}{c} q_{2s-2n-4}}{q_{2n-2s-3} - \frac{d}{c} q_{2s-2n-4}} = \frac{p_{2s-2n-3-2j}}{q_{2s-2n-3-2j}} = \frac{p_{2(s-n-2)+1-2j}}{q_{2(s-n-2)+1-2j}}.$$

■

Lemma 4

$$R_{n+1} < R_n$$

PROOF. The statement of the lemma is equivalent to

$$(-1)^n (d q_n q_{n+1} - p_n p_{n+1}) > 0. \qquad (8)$$

If n is even, then $\frac{p_n}{q_n} < \sqrt{d}$ and $\frac{p_{n+1}}{q_{n+1}} > \sqrt{d}$. Furthermore, since $\frac{p_{n+1}}{q_{n+1}} - \sqrt{d} < \sqrt{d} - \frac{p_n}{q_n}$, we have $\frac{p_n}{q_n} + \frac{p_{n+1}}{q_{n+1}} < 2\sqrt{d}$. Therefore

$$\frac{p_n}{q_n} \cdot \frac{p_{n+1}}{q_{n+1}} < \left[\left(\frac{p_n}{q_n} + \frac{p_{n+1}}{q_{n+1}} \right) \Big/ 2 \right]^2 < d$$

and inequality (8) is satisfied. If n is odd, the proof is completely analogous.

■

Proposition 1 *If d is a square-free positive integer such that $s(d) > 2$, then*

$$|j(d,n)| \le \frac{s(d)-3}{2}$$

for all $n \ge 0$.

PROOF. According to Lemma 3 it suffices to consider the case $j > 0$. Let $R_n = \frac{p_{2n+1+2j}}{q_{2n+1+2j}}$. By Lemma 2 there is no loss of generality in assuming that $n < s$.

Assume first that s is even, say $s = 2t$. Then $R_{t-1} = \frac{p_{s-1}}{q_{s-1}}$ and $R_{s-1} = \frac{p_{2s-1}}{q_{2s-1}}$. If $n < t-1$, then Lemma 4 clearly implies that $2n+1+2j \le s-2$ and $2j \le s-3$. Since s is even, we have $j \le \frac{s-4}{2}$. For $n = t-1$ or $n = s-1$ we obtain $j = 0$. If $t-1 < n < s-1$, then $2n+1+2j \le 2s-2$ and $2j \le 2s-3-2n \le s-3$. Thus we have again $j \le \frac{s-4}{2}$.

Assume now that s is odd, say $s = 2t+1$. Instead of applying Newton's method for $x_0 = \frac{p_{t-1}}{q_{t-1}}$, we will apply "regula falsi" method for $x_0 = \frac{p_{t-1}}{q_{t-1}}$ and $x_1 = \frac{p_t}{q_t}$. It was proved by Frank [1962] that with this choice of x_0 and x_1 we have

$$R_{t-1,t} = \frac{x_0 \cdot x_1 + d}{x_0 + x_1} = \frac{p_{s-1}}{q_{s-1}}\,.$$

If $t-1 < n < s-1$, then from $R_{s-1} = \frac{p_{2s-1}}{q_{2s-1}}$ we obtain that $j \le \frac{s-3}{2}$ as above. Thus, assume that $n \le t-1$. Since the number $\frac{x_0x_1+d}{x_0+x_1}$ lies between the numbers x_0 and x_1, we conclude that

$$|R_{t-1,t} - \sqrt{d}| < |R_{t-1} - \sqrt{d}|.$$

Hence, by Lemma 4, we have $2n+1+2j \le s-2$ and $j \le \frac{s-3}{2}$. ∎

The following lemma shows that the estimate from Proposition 1 is sharp.

Lemma 5 *Let $t \ge 1$ and $m \ge 5$ be integers such that $m \equiv \pm 1 \pmod 6$ and let $d = F_{m-2}^2[(2F_{m-2}t - F_{m-4})^2+4]/4$. Then*

$$\sqrt{d} = [\frac{1}{2}F_{m-2}(2F_{m-2}t - F_{m-4}); \overline{2t-1, \underbrace{1,1,\ldots,1,1}_{m-3}, 2t-1, F_{m-2}(2F_{m-2}t - F_{m-4})}]. \quad (9)$$

Therefore, $s(d) = m$.

Furthermore, $R_0 = \frac{p_{m-2}}{q_{m-2}}$ and hence $j(d,0) = \frac{m-3}{2}$, $j(d,km) = \frac{m-3}{2}$ and $j(d,km-2) = -\frac{m-3}{2}$ for $k \ge 1$.

PROOF. Since $m \equiv \pm 1 \pmod 6$, the number $\frac{1}{2}F_{m-2}F_{m-4}$ is an integer. It is clear that $a_0 = \lfloor \sqrt{d} \rfloor = \frac{1}{2}F_{m-2}(2F_{m-2}t - F_{m-4})$. Then we have

$$\begin{aligned} a_1 &= \left\lfloor \frac{1}{\sqrt{d} - a_0} \right\rfloor = \left\lfloor \frac{\sqrt{d} + a_0}{d - a_0^2} \right\rfloor = \left\lfloor \frac{\sqrt{d} + a_0}{F_{m-2}^2} \right\rfloor = \left\lfloor \frac{2a_0}{F_{m-2}^2} \right\rfloor = \left\lfloor 2t - \frac{F_{m-4}}{F_{m-2}} \right\rfloor \\ &= 2t - 1. \end{aligned}$$

Let

$$\sqrt{d} = a_0 + \frac{1}{a_1 + \frac{1}{\alpha_2}}\,.$$

Then

$$\frac{1}{\alpha_2} = \frac{\sqrt{d} - a_0 + F_{m-2}F_{m-3}}{F_{m-2}^2}$$

and

$$\frac{1}{\alpha_2} > \frac{F_{m-3}}{F_{m-2}}\,. \tag{10}$$

Since

$$\begin{aligned} \sqrt{d} &= \sqrt{a_0^2 + F_{m-2}^2} = a_0\sqrt{1 + \frac{F_{m-2}^2}{a_0^2}} < a_0 + \frac{F_{m-2}^2}{2a_0} \le a_0 + \frac{F_{m-2}^2}{F_{m-2}F_{m-1}} \\ &= a_0 + \frac{F_{m-2}}{F_{m-1}} \end{aligned}$$

we have

$$\frac{1}{\alpha_2} < \frac{\frac{F_{m-2}^2}{F_{m-1}} + F_{m-2}F_{m-3}}{F_{m-2}^2} = \frac{F_{m-1}F_{m-3} + 1}{F_{m-1}F_{m-2}} = \frac{F_{m-2}}{F_{m-1}}\,. \tag{11}$$

From inequalities (10) and (11) we conclude that

$$\frac{1}{\alpha_2} = [0; \underbrace{1, 1, \ldots, 1}_{m-3}, y] \tag{12}$$

and $a_2 = a_3 = \cdots = a_{m-2} = 1$. Furthermore, from (12) we have

$$\frac{1}{\alpha_2} = \frac{yF_{m-3} + F_{m-4}}{yF_{m-2} + F_{m-3}}$$

and

$$\begin{aligned} y &= \frac{\alpha_2 F_{m-4} - F_{m-3}}{F_{m-2} - \alpha_2 F_{m-3}} \\ &= \frac{F_{m-2} + F_{m-3}a_0 - F_{m-3}\sqrt{d}}{F_{m-2}(\sqrt{d} - a_0)} \cdot \frac{\sqrt{d} + a_0}{\sqrt{d} + a_0} \cdot \frac{F_{m-2} + F_{m-3}a_0 + F_{m-3}\sqrt{d}}{F_{m-2} + F_{m-3}a_0 + F_{m-3}\sqrt{d}} \\ &= \frac{\sqrt{d} + a_0}{F_{m-2}[F_{m-2} + F_{m-3}(\sqrt{d} + a_0)]} \cdot [1 + F_{m-3}F_{m-2}(2t-1)] \,. \end{aligned} \tag{13}$$

Let $\frac{1}{z} = y - (2t-1)$. From (13) we obtain

$$z = \frac{F_{m-2}^2 + F_{m-2}F_{m-3}(\sqrt{d} + a_0)}{\sqrt{d} - a_0 + F_{m-2}F_{m-3}} > \frac{2a_0 F_{m-2}F_{m-3}}{1 + F_{m-2}F_{m-3}} \geq \frac{4}{3}a_0 \geq a_0 + 1 \,.$$

We have $a_{m-1} = \lfloor y \rfloor = 2t - 1$ and $a_m \geq a_0 + 1$. But now from [Perron 1954, Satz 3.13] it follows that $a_m = 2a_0$ and $s(d) = m$.

Let us consider now the approximant

$$\begin{aligned} R_0 &= \frac{1}{2}\Big(a_0 + \frac{d}{a_0}\Big) = \frac{a_0^2 + d}{2a_0} = \frac{2d - F_{m-2}^2}{F_{m-2}(2F_{m-2}t - F_{m-4})} \\ &= \frac{F_{m-2}[(2F_{m-2}t - F_{m-4})^2 + 2]}{2(2F_{m-2}t - F_{m-4})} \,. \end{aligned}$$

From (9) we have

$$\begin{aligned} \frac{p_{m-2}}{q_{m-2}} &= a_0 + \frac{1}{a_1 + \frac{F_{m-3}}{F_{m-2}}} = a_0 + \frac{F_{m-2}}{(2t-1)F_{m-2} + F_{m-3}} \\ &= a_0 + \frac{F_{m-2}}{2tF_{m-2} - F_{m-4}} = R_0 \,, \end{aligned}$$

and $j(d,0) = \frac{m-3}{2}$ as we claimed. Now Lemmas 2 and 3 imply that $j(d,km) = \frac{m-3}{2}$ and $j(d,km-2) = -\frac{m-3}{2}$ for $k \geq 1$. ■

Corollary 1

$$\sup\{|j(d,n)|\} = +\infty$$

$$\limsup \Big\{ \frac{|j(d,n)|}{s(d)} \Big\} = \frac{1}{2}$$

It remains the question how large can be $|j|$ compared with d. In [Cohn 1977] it was proved that $s(d) < \frac{7}{2\pi^2}\sqrt{d}\log d + \mathcal{O}(\sqrt{d})$. However, under the extended Riemann Hypothesis for $\mathbf{Q}(\sqrt{d})$ one would expect that $s(d) = \mathcal{O}(\sqrt{d}\log\log d)$ (see [Williams 1981; Patterson and Williams 1985]) and therefore $|j(d,n)| = \mathcal{O}(\sqrt{d}\log\log d)$.

Let

$$d(j) = \min\{d : \text{there exist } n \text{ such that } j(d,n) \geq j\}.$$

In Table 1 we list values of $d(j)$ for $1 \leq j \leq 48$ such that $d(j) > d(j')$ for $j' < j$. We also give corresponding values n and k such that $R_n = \frac{p_k}{q_k} = \frac{p_{2n+1+2j}}{q_{2n+1+2j}}$.

We don't have enough data to support any conjecture about the rate of growth of $d(j)$. In particular, it remains open whether $\limsup\{\frac{|j(d,n)|}{\sqrt{d}}\} > 0$.

3 Number of good approximants

Proposition 2 *If $a_{n+1} > 2\sqrt{\sqrt{d}+1}$, then R_n is a convergent of $\sqrt{d}$.*

Proof. From (2) and Lemma 1 we have

$$R_n - \sqrt{d} < \frac{1}{2p_n q_n^3 a_{n+1}^2}\,.$$

Let $R_n = \frac{u}{v}$, where $(u,v) = 1$. Then certainly $v \leq 2p_n q_n$, and

$$|\sqrt{d} - \frac{u}{v}| < \frac{1}{8p_n^2 q_n^2} \cdot \frac{4p_n}{q_n a_{n+1}^2} < \frac{1}{2v^2} \cdot \frac{1}{\sqrt{d}+1} \cdot \Big(\sqrt{d} + \frac{1}{a_{n+1}q_n^2}\Big) < \frac{1}{2v^2}\,,$$

which proves the proposition. ■

Theorem 2 *R_n is a convergent of $\sqrt{d}$ for all $n \geq 0$ if and only if $s(d) \leq 2$.*

Proof. As we mentioned in the introduction, the result of Mikusiński [1954] imply that if $s(d) \leq 2$, then all R_n are convergents of $\sqrt{d}$.

Assume now that R_n is a convergent of $\sqrt{d}$ for all $n \geq 0$. Then we must have $R_n = \frac{p_{2n+1}}{p_{2n+1}}$ for all $n \geq 0$. Indeed, this is a consequence of the fact that $R_{s-1} = \frac{p_{2s-1}}{q_{2s-1}}$, Corollary 1 and Lemma 4. Therefore, $R_0 = \frac{p_1}{q_1}$ and

$$R_{ks-1} = \frac{p_{2ks+1}}{q_{2ks+1}} \tag{14}$$

$d(j)$	$s(d)$	n	k	$j(d,n)$	$\log d(j)/\log j(d,n)$	$\sqrt{d(j)}/j(d,n)$
13	5	5	3	1		3.60555
124	16	1	7	2	6.95420	5.56776
181	21	4	15	3	4.73188	4.48454
989	32	7	23	4	4.97491	7.86209
1021	49	12	35	5	4.30494	6.39062
1549	69	18	49	6	4.09953	6.55956
3277	35	6	27	7	4.15984	8.17787
3949	128	79	175	8	3.98242	7.85513
10684	212	46	113	10	4.02873	10.3363
12421	121	30	89	14	3.57216	7.96068
22081	218	62	155	15	3.69361	9.90645
33619	282	83	199	16	3.75925	11.4597
39901	449	287	609	17	3.73927	11.7501
45109	470	143	325	19	3.63969	11.1784
48196	374	129	299	20	3.59946	10.9768
60631	504	149	343	22	3.56273	11.1924
78439	696	208	467	25	3.50125	11.2028
81841	494	153	361	27	3.43237	10.5955
170689	743	207	473	29	3.57783	14.2464
179356	776	500	1063	31	3.52276	13.6614
194374	738	220	505	32	3.51370	13.7775
224239	1008	302	673	34	3.49382	13.9276
238081	979	613	1297	35	3.48218	13.9410
241021	1008	311	695	36	3.45823	13.6372
242356	1090	710	1499	39	3.38418	12.6230
253324	984	291	667	42	3.32893	11.9836

Table 1: $d(j)$ for $1 \leq j \leq 42$

for all $n \geq 0$. Let $\sqrt{d} = [a_0; \overline{a_1, \ldots, a_{s-1}, 2a_0}]$ and $d = a_0^2 + t$. Then

$$R_{ks} = \frac{\alpha p_{2ks+2} + p_{2ks+1}}{\alpha q_{2ks+2} + q_{2ks+1}}, \tag{15}$$

where

$$\alpha = \frac{2a_0 - a_1 t}{(a_1 a_2 + 1)t - 2a_0}$$

(see [Komatsu 1999, Corollary 1]). From (14) and (15) it follows that $\alpha = 0$ and therefore $t = \frac{2a_0}{a_1}$. It is well known (see e.g. [Sierpiński 1987, p. 322]) that if $d = a_0^2 + t$, where t is a divisor of $2a_0$, then $s(d) \leq 2$. ■

If R_n is a convergent of $\sqrt{d}$, then we will say that R_n is a "good approximant". Let

$$b(d) = |\{n \,:\, 0 \leq n \leq s-1 \text{ and } R_n \text{ is a convergent of } \sqrt{d}\}|.$$

Theorem 2 shows that if $s(d) > 2$ then $\frac{s(d)}{b(d)} > 1$. Komatsu [1999] proved that if $d = (2x+1)^2 + 4$ then $b(d) = 3$, $s(d) = 5$ (see also [Elezović 1997]) and if $d = (2x+3)^2 - 4$ then $b(d) = 4$, $s(d) = 6$.

Example 1 If $d = 16x^4 - 16x^3 - 12x^2 + 16x - 4$, where $x \geq 2$, then $s(d) = 8$ and $b(d) = 6$. Using algorithm (1) it is straightforward to check that

$$\sqrt{d} = [(2x+1)(2x-2); \overline{x, 1, 1, 2x^2 - x - 2, 1, 1, x, 2(2x+1)(2x-2)}]\,.$$

Hence, $s(d) = 8$.

Now the direct computation shows that

$$\begin{aligned}
R_0 &= \frac{p_3}{q_3} = \frac{2x(4x^2-3)}{2x+1} \\
R_1 &= \frac{p_5}{q_5} = \frac{(2x-1)(8x^4-8x^2+1)}{2x(2x^2-1)} \\
R_3 &= \frac{p_7}{q_7} = \frac{(2x^2-1)(16x^4-16x^2+1)}{x(2x+1)(4x^2-3)} \\
R_5 &= \frac{p_9}{q_9} = \frac{(2x-1)(128x^8-256x^6+160x^4-32x^2+1)}{4x(2x^2-1)(8x^4-8x^2+1)} \\
R_6 &= \frac{p_{11}}{q_{11}} = \frac{2x(4x^2-3)(64x^6-96x^4+36x^2-3)}{(2x+1)(8x^3-6x-1)(8x^3-6x+1)} \\
R_7 &= \frac{p_{15}}{q_{15}} = \frac{(8x^4-8x^2+1)(256x^8-512x^6+320x^4-64x^2+1)}{2x(2x+1)(2x^2-1)(4x^2-3)(16x^4-16x^2+1)}\,.
\end{aligned}$$

Hence, $b(d) = 6$.

In the same manner we can check that for $d = 16x^4 + 48x^3 + 52x^2 + 32x + 12$, $x \geq 1$, we have also $s(d) = 8$ and $b(d) = 6$.

Let

$$s_b = \min\{s : \text{there exist } d \text{ such that } s(d) = s \text{ and } b(d) = b\}.$$

We know that $s_1 = 1$, $s_2 = 2$, $s_3 = 5$, $s_4 = 6$ and $s_6 = 8$. In Table 2 we list upper bounds for s_b obtained by experiments.

b	$s_b \leq$	$s_b/b \leq$	b	$s_b \leq$	$s_b/b \leq$
3	5	1.66667	17	43	2.52941
4	6	1.50000	18	32	1.77778
5	9	1.80000	19	41	2.15789
6	8	1.33333	20	34	1.70000
7	13	1.85714	21	41	1.95238
8	12	1.50000	22	46	2.09091
9	17	1.88889	23	69	3.00000
10	14	1.40000	24	38	1.58333
11	23	2.09091	25	69	2.76000
12	18	1.50000	26	50	1.92308
13	27	2.07692	27	97	3.59259
14	22	1.57143	28	58	2.07143
15	41	2.73333	29	97	3.34483
16	26	1.62500	30	58	1.93333

Table 2: upper bounds for s_b

Questions: Is it true that $\inf\{s_b/b : b \geq 3\} = \frac{4}{3}$?
What can be said about $\sup\{s_b/b : b \geq 1\}$?

Example 2 Let $d = 25[(10x+1)^2 + 4]$. Then

$$\begin{aligned}\sqrt{d} = [50x+5; &\overline{x, 9, 1, x-1, 4, 1, 4x-1, 1, 1, 1, 1, 1, x-1, 1, 1, 25x+2,}\\ &\overline{4x, 2, 2, x-1, 1, 1, 2, 2, 1, x-1, 2, 2, 4x, 25x+2, 1, 1, x-1,}\\ &\overline{1, 1, 1, 1, 4x-1, 1, 4, x-1, 1, 9, x, 100x+10}] .\end{aligned}$$

Hence, $s(d) = 43$. Furthermore, $b(d) \geq 15$. Indeed, it may be verified that $R_n = \frac{p_k}{q_k}$ for $(n, k) \in \{(0, 3), (3, 11), (6, 15), (11, 23), (14, 27), (15, 35), (18, 41), (23, 43), (26, 49), (27, 57), (30, 61), (35, 69), (38, 73), (41, 81), (42, 85)\}$.

We expect that Example 2 may be generalized to yield positive integers d with $b(d)$ arbitrary large. In this connection, we have the following conjecture.

Conjecture 1 *Let* $d = F_m^2[(2F_m x \pm F_{m-3})^2 + 4]$, *where* $m \equiv \pm 1 \pmod 6$. *Then* $b(d) \geq 3F_m$.

We have checked Conjecture 1 for $m \leq 25$. We have also the more precise form of Conjecture 1. Namely, we have noted that if $d = F_m^2[(2F_m x + F_{m-3})^2 + 4]$, where x is sufficiently large, then in the sequence $a_1, a_2, \ldots, a_{s-1}$ the numbers $x - 1$, x, $4x - 1$ and $4x$ appear $2F_n - F_{n-3} - 3$, $F_{n-3} + 2$, $L_{n-3} + 1$ and $2F_{n-3}$ times, respectively, and the number $\frac{a_0 - 1}{2}$ appears once. If this conjecture on the sequence $a_1, a_2, \ldots, a_{s-1}$ is true, then at least $3F_n$ elements in that sequence are greater then $2\sqrt{\sqrt{d} + 1}$, and Proposition 2 implies $b(d) \geq 3F_n$. We have also noted similar phenomena for $d = F_m^2[(2F_m x - F_{m-3})^2 + 4]$.

As in the case of $j(d, n)$, we are also interested in the question how large can be $b(d)$ compared with d. Let

$$d_b = \min\{d \,:\, b(d) \geq b\}.$$

In Table 3 we listed values of d_b for $1 \leq b \leq 102$ such that $d_b > d_{b'}$ for $b' < b$.

Consider the expression $\frac{\log d_b}{\log b}$. Conjecture 1 implies that

$$\sup\left\{\frac{\log d_b}{\log b} \,:\, b \geq 2\right\} \leq 4$$

and Table 3 suggests that this bound might be less than 4. It would be interesting to find exact value for $\sup\{\frac{\log d_b}{\log b} \,:\, b \geq 2\}$.

d_b	$s(d_b)$	b	$\log d_b/\log b$
2	1	1	
3	2	2	1.58496
13	5	3	2.33472
21	6	4	2.19616
43	10	6	2.09917
76	12	8	2.08264
244	26	14	2.08300
796	44	16	2.40916
1141	58	18	2.43556
1516	76	20	2.44475
2629	100	22	2.54748
3004	108	24	2.51969
3949	128	26	2.54173
4204	116	28	2.50399
6589	134	30	2.58531
10021	190	32	2.65815
12229	174	36	2.62635
18484	258	38	2.70087
19996	272	40	2.68463
22309	250	42	2.67887
23149	288	50	2.56893
31669	368	52	2.62274
46981	430	58	2.64934
52789	514	62	2.63477
73516	644	64	2.69430
76549	548	68	2.66517
87109	648	72	2.65976
103741	618	74	2.65100
140701	690	80	2.70523
163669	776	82	2.72439
180709	954	86	2.71749
228229	1160	90	2.74192
249601	950	92	2.74839
273361	1076	94	2.75539
279301	1214	98	2.73503
344509	1164	102	2.75675

Table 3: d_b for $b \leq 102$

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