

RANKS AND INTEGER POINTS ON ELLIPTIC CURVES INDUCED BY FIBONACCI TRIPLES

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Dedicated to the memory of my dear friend and coauthor Florian Luca

ABSTRACT. Let F_n and L_n denote the Fibonacci and Lucas numbers, respectively, and consider

$$E_k : y^2 = (F_{2k}x + 1)(F_{2k+2}x + 1)(F_{2k+4}x + 1).$$

These elliptic curves arise naturally from the regular Diophantine triples

$$\{F_{2k}, F_{2k+2}, F_{2k+4}\}.$$

For odd k , we exhibit the rational point

$$Q_k = \left(-\frac{F_{k-1}}{L_k F_{k+1} F_{k+2}}, \frac{F_{2k+1}}{L_k F_{k+1} F_{k+2}} \right).$$

For every odd $k \geq 3$, this point is independent of the standard point $P_k = (0, 1)$; in particular, $\text{rank } E_k(\mathbb{Q}) \geq 2$. Moreover, if $k \geq 3$ is odd and $\text{rank } E_k(\mathbb{Q}) = 2$, then all integer points on E_k are exactly the points arising from the two known solutions of the Hoggatt-Bergum extension problem. By parametrizing the two conics $L^2 - 5F^2 = \pm 4$ and applying an injective specialization criterion, we also show that the corresponding one-parameter elliptic families have generic ranks 2 in the odd case and 1 in the even case. Finally, we discuss computational data and propose the heuristic rank distribution $1/4, 1/2, 1/4$ for ranks 1, 2, 3, respectively, with density zero for rank at least 4.

1. INTRODUCTION

A set of positive integers $\{a_1, \dots, a_m\}$ is called a Diophantine m -tuple if $a_i a_j + 1$ is a perfect square for all $i \neq j$. The Fibonacci numbers satisfy

$$F_{2k} F_{2k+2} + 1 = F_{2k+1}^2, \quad F_{2k} F_{2k+4} + 1 = F_{2k+2}^2,$$

and

$$F_{2k+2} F_{2k+4} + 1 = F_{2k+3}^2.$$

Hence

$$\{F_{2k}, F_{2k+2}, F_{2k+4}\}$$

is a Diophantine triple for every positive integer k . Hoggatt and Bergum [11] observed that it can be extended to the Diophantine quadruple

$$\{F_{2k}, F_{2k+2}, F_{2k+4}, 4F_{2k+1} F_{2k+2} F_{2k+3}\},$$

and conjectured that the fourth element is unique. This conjecture was proved in [2]. A simple consequence of this result is that if $\{F_{2k}, F_{2k+2}, F_{2k+4}, d\}$

is a Diophantine quadruple, then d cannot be a Fibonacci number (see [5, Section 4.9.1]).

Several authors considered the question of how large Diophantine tuples consisting of Fibonacci numbers can be. He, Luca, and Togbé [10] proved that if $\{F_{2n}, F_{2n+2}, F_k\}$ is a Diophantine triple, then $k = 2n+4$ or $k = 2n-2$ (when $n > 1$), except when $n = 2$, in which case $k = 1$ is also possible. Fujita and Luca [7] proved that there are only finitely many Diophantine quadruples consisting of Fibonacci numbers, and in [8] they proved that there are no such quadruples.

To the triple $\{F_{2k}, F_{2k+2}, F_{2k+4}\}$ one associates the elliptic curve

$$(1) \quad E_k : \quad y^2 = (F_{2k}x + 1)(F_{2k+2}x + 1)(F_{2k+4}x + 1).$$

We say that E_k is induced by the triple $\{F_{2k}, F_{2k+2}, F_{2k+4}\}$. The curves E_k were studied in [4] and, more recently, in [5, Sections 4.8-4.9]. It was proved that

$$E_k(\mathbb{Q})_{\text{tors}} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

and that $\text{rank } E_k(\mathbb{Q}) \geq 1$. Under the assumption that $\text{rank } E_k(\mathbb{Q}) = 1$, all integer points were determined. In particular, for $k \geq 2$ they are

$$(2) \quad (0, \pm 1)$$

and

$$(3) \quad (d_k, \pm(2F_{2k+1}F_{2k+2} - 1)(2F_{2k+2}^2 + 1)(2F_{2k+2}F_{2k+3} + 1)),$$

where

$$d_k = 4F_{2k+1}F_{2k+2}F_{2k+3}.$$

Computations with PARI/GP suggested an unusual rank distribution in this family: among the curves of odd expected rank, curves of expected rank 3 occur almost as frequently as curves of expected rank 1; see [5, Remark 4.9.5]. The purpose of this note is to explain this phenomenon. The main new ingredient is a second systematic rational point which appears precisely in the odd-index subfamily.

Our main result is the following.

Theorem 1.1. *Let $k \geq 3$ be odd. Then:*

(i) *the point*

$$Q_k = \left(-\frac{F_{k-1}}{L_k F_{k+1} F_{k+2}}, \frac{F_{2k+1}}{L_k F_{k+1} F_{k+2}} \right)$$

belongs to $E_k(\mathbb{Q})$;

(ii) *the points $P_k = (0, 1)$ and Q_k are independent, and therefore*

$$\text{rank } E_k(\mathbb{Q}) \geq 2;$$

(iii) *if $\text{rank } E_k(\mathbb{Q}) = 2$, then the integer points on E_k are precisely the points in (2) and (3).*

For $k = 1$ the same formula gives $Q_1 = (0, 1) = P_1$, so the restriction $k \geq 3$ in parts (ii) and (iii) is necessary.

2. THE ADDITIONAL POINT

We use the standard notation

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n,$$

and

$$L_0 = 2, \quad L_1 = 1, \quad L_{n+2} = L_{n+1} + L_n.$$

We shall repeatedly use

$$L_n = F_{n-1} + F_{n+1}, \quad F_{2n} = F_n L_n,$$

$$F_{2n+1} = F_n^2 + F_{n+1}^2,$$

and Cassini's identity

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n.$$

For odd k , put

$$f = F_k, \quad u = F_{k+1}, \quad v = F_{k+2},$$

$$\ell = L_k, \quad \ell_1 = L_{k+1}, \quad \ell_2 = L_{k+2}, \quad r = F_{2k+1}.$$

We shall also write

$$a = F_{2k}, \quad b = F_{2k+2}, \quad c = F_{2k+4}.$$

Notice that

$$(4) \quad b - a = r, \quad c = a + b + 2r,$$

so the Fibonacci triple is a regular Diophantine triple (see e.g. [5, Section 1.1]).

The following three identities will be useful:

$$(5) \quad uv - fF_{k-1} = r,$$

$$(6) \quad \ell v - \ell_1 F_{k-1} = r,$$

$$(7) \quad \ell u - \ell_2 F_{k-1} = 1.$$

They follow immediately from the standard addition formulas for Fibonacci and Lucas numbers (or directly from Cassini's identity and the recurrence).

Proposition 2.1. *Let k be odd and let*

$$D_k = L_k F_{k+1} F_{k+2}.$$

Then

$$Q_k = \left(-\frac{F_{k-1}}{D_k}, \frac{F_{2k+1}}{D_k} \right)$$

is a rational point on E_k .

Proof. Put $q = -F_{k-1}/D_k$. Since $a = f\ell$, $b = u\ell_1$ and $c = v\ell_2$, identities (5)-(7) give

$$(8) \quad aq + 1 = \frac{r}{uv},$$

$$(9) \quad bq + 1 = \frac{r}{v\ell},$$

$$(10) \quad cq + 1 = \frac{1}{u\ell}.$$

Therefore

$$(aq + 1)(bq + 1)(cq + 1) = \frac{r^2}{\ell^2 u^2 v^2} = \left(\frac{r}{D_k} \right)^2,$$

which proves the assertion. $\square$

For example,

$$Q_3 = \left(-\frac{1}{60}, \frac{13}{60} \right), \quad Q_5 = \left(-\frac{3}{1144}, \frac{89}{1144} \right).$$

3. INDEPENDENCE OF THE TWO POINTS

We follow the 2-descent approach used in [4, 6, 3]. Applying $X = abcx$ and $Y = abcy$ transforms (1) into

$$(11) \quad E'_k : \quad Y^2 = (X + bc)(X + ac)(X + ab).$$

The three non-trivial points of order 2 are

$$A = (-bc, 0), \quad B = (-ac, 0), \quad C = (-ab, 0),$$

and $P_k = (0, 1)$ corresponds to

$$P = (0, abc).$$

We use the same letters for a point on E_k and its image on E'_k when no confusion can arise.

For $R = (X, Y) \in E'_k(\mathbb{Q})$, the usual 2-descent maps are represented, away from the corresponding 2-torsion point, by the square classes of

$$X + bc, \quad X + ac, \quad X + ab.$$

In particular, $R \in 2E'_k(\mathbb{Q})$ if and only if all three corresponding classes are trivial, with the standard special definitions at the 2-torsion points; cf. [4].

We need two elementary coprimality observations.

Lemma 3.1. *Let $k \geq 3$ be odd.*

(i) *If an odd prime p divides $v = F_{k+2}$, then*

$$p \nmid r f u \ell_1 \ell_2 (c - a)(c - b).$$

(ii) *If an odd prime p divides $\ell_2 = L_{k+2}$, then*

$$p \nmid r f u v \ell \ell_1 (c - a)(c - b).$$

Proof. Assume first that $p \mid v$. Since $v = f + u$, we have $u \equiv -f \pmod{p}$. Cassini's identity, in the form

$$u^2 - fu - f^2 = -1,$$

gives $f^2 \equiv -1 \pmod{p}$. Consequently

$$r = f^2 + u^2 \equiv -2, \quad \ell = 2u - f \equiv -3f,$$

$$\ell_1 = f + v \equiv f, \quad \ell_2 = 2u + v \equiv -2f \pmod{p}.$$

Furthermore,

$$c - a = L_{2k+2} = \ell_1^2 - 2 \equiv -3 \pmod{p}$$

and

$$c - b = F_{2k+3} = u^2 + v^2 \equiv -1 \pmod{p}.$$

This proves (i).

Now assume $p \mid \ell_2$. Since $\ell_2 = f + 3u$, we have $f \equiv -3u \pmod{p}$. Cassini's identity gives $5u^2 \equiv 1 \pmod{p}$. Hence

$$r = f^2 + u^2 \equiv 2, \quad v = f + u \equiv -2u,$$

$$\ell = 2u - f \equiv 5u, \quad \ell_1 = 2f + u \equiv -5u \pmod{p}.$$

Also

$$c - a = L_{2k+2} = \ell_1^2 - 2 \equiv 3 \pmod{p},$$

and

$$c - b = F_{2k+3} = u^2 + v^2 \equiv 1 \pmod{p}.$$

This proves (ii). $\square$

We shall use the classical theorem of Cohn [1]: the only square Fibonacci numbers are $F_1 = F_2 = 1$ and $F_{12} = 144$, while the only Fibonacci numbers which are twice a square are $F_3 = 2$ and $F_6 = 8$. Similarly, the only square Lucas numbers are $L_1 = 1$ and $L_3 = 4$.

We shall also use the elementary parity facts

$$(12) \quad v_2(F_n) = \begin{cases} 0, & 3 \nmid n, \\ 1, & 3 \mid n, \end{cases} \quad v_2(L_n) = \begin{cases} 0, & 3 \nmid n, \\ 2, & 3 \mid n, \end{cases} \quad (n \text{ odd}),$$

which follow immediately from the Fibonacci and Lucas recurrences modulo 4 and 8, respectively. Thus, for odd n , if every odd prime occurs to even exponent in F_n (respectively L_n), then F_n is a square or twice a square (respectively L_n is a square).

The following table is obtained by applying the three descent homomorphisms to Q , $Q + P$, $Q + C$ and $Q + P + C$. Here $\square$ denotes the square of a rational number.

R	$ax + 1$	$bx + 1$	$cx + 1$
Q	$ruv \square$	$rv\ell \square$	$u\ell \square$
$Q + P$	$r\ell_1\ell_2 \square$	$rf\ell_2 \square$	$f\ell_1 \square$
$Q + C$	$rv\ell_1(c - a) \square$	$rfv(c - b) \square$	$f\ell_1(c - a)(c - b) \square$
$Q + P + C$	$ru\ell_2(c - a) \square$	$r\ell\ell_2(c - b) \square$	$u\ell(c - a)(c - b) \square$

For the four points

$$Q + A, \quad Q + B, \quad Q + P + A, \quad Q + P + B,$$

exactly two of the three corresponding coefficients are negative.

Lemma 3.2. *Let $k \geq 3$ be odd and let $T \in E'_k(\mathbb{Q})_{\text{tors}} = \{O, A, B, C\}$. Then*

$$Q + T \notin 2E'_k(\mathbb{Q}), \quad Q + P + T \notin 2E'_k(\mathbb{Q}).$$

Proof. The descent classes of a point R are represented by the coefficients in the row corresponding to $R + P$ in (13), because the descent classes of P are represented by bc, ac, ab .

For $T = A$ or B , the assertion follows immediately from the two negative coefficients mentioned above.

Suppose $T = O$ or C . If $Q + T \in 2E'_k(\mathbb{Q})$, the corresponding row $Q + P + T$ shows, by Lemma 3.1(ii), that every odd prime occurs to an even exponent in L_{k+2} . By (12), L_{k+2} is a square, contrary to Cohn's theorem, since $k + 2 \geq 5$ is odd.

Similarly, if $Q + P + T \in 2E'_k(\mathbb{Q})$, the row belonging to $Q + T$ and Lemma 3.1(i) imply

$$F_{k+2} = \square \quad \text{or} \quad F_{k+2} = 2\square,$$

again impossible. $\square$

Proposition 3.3. *For every odd $k \geq 3$, the points P_k and Q_k are independent in $E_k(\mathbb{Q})$.*

Proof. It was proved in [4] that

$$E_k(\mathbb{Q})_{\text{tors}} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

Moreover, [4, Lemma 2] shows that

$$P + T \notin 2E'_k(\mathbb{Q})$$

for every torsion point T , except possibly in the exceptional case in which $c, c - a$ and $c - b$ are all twice squares. Here $c = F_{2k+4}$, and Cohn's theorem excludes this possibility for $k \geq 3$.

Together with Lemma 3.2, this shows that no non-zero element among

$$P + T, \quad Q + T, \quad P + Q + T, \quad T \in E'_k(\mathbb{Q})_{\text{tors}},$$

belongs to $2E'_k(\mathbb{Q})$.

Suppose that $mP + nQ$ is torsion. If m and n are not both even, reduction modulo $2E'_k(\mathbb{Q})$ contradicts the preceding paragraph. Hence m and n are both even, say $m = 2m_1$ and $n = 2n_1$. Since $2(m_1P + n_1Q)$ is torsion, the point $m_1P + n_1Q$ is torsion as well. Repeating the same argument shows that m and n are divisible by arbitrarily high powers of 2. Therefore $m = n = 0$. $\square$

Together with Proposition 2.1, this proves parts (i) and (ii) of Theorem 1.1.

4. INTEGER POINTS WHEN THE RANK IS TWO

We now assume throughout this section that $k \geq 3$ is odd and

$$\text{rank } E_k(\mathbb{Q}) = 2.$$

The proof of Proposition 3.3 shows that the images of P and Q are linearly independent in $E'_k(\mathbb{Q})/(2E'_k(\mathbb{Q}) + E'_k(\mathbb{Q})_{\text{tors}})$. Since $\text{rank } E'_k(\mathbb{Q}) = 2$, these images form a basis. Therefore, every point of E'_k is congruent modulo $2E'_k(\mathbb{Q})$ to one of the sixteen points

$$T, \quad P + T, \quad Q + T, \quad P + Q + T, \quad T \in \{O, A, B, C\}.$$

This is exactly the same enlargement from eight to sixteen 2-descent classes which occurs in the rank-2 arguments of [6, 3].

For the eight classes not involving Q , the class-by-class elimination in the proof of [4, Theorem 4], specialized to the Fibonacci family as in [4, Theorem 6], shows that no integral point can occur outside the class $P + 2E'_k(\mathbb{Q})$. The possible square or twice-square exceptional Fibonacci terms are handled there by Cohn's theorem (with the remaining case $k = 4$ eliminated modulo 3); in particular, no exception occurs for odd $k \geq 3$.

Only the local eliminations attached to the eight classes T and $P + T$ are being reused from the proof of [4, Theorem 4]; those arguments depend on the square-class equations for the given coset, not on the global rank-one hypothesis used there to exhaust all cosets. It remains to eliminate the eight new classes involving Q .

Lemma 4.1. *None of the eight classes*

$$Q + T, \quad Q + P + T, \quad T \in \{O, A, B, C\},$$

contains an integer point on E_k .

Proof. For $T = A$ or B , exactly two coefficients in the corresponding descent system are negative, hence there can be no integral solution. Indeed, since $a, b, c > 1$, for integral x , all three $ax + 1$, $bx + 1$, $cx + 1$ are positive when $x \geq 0$ and all are negative when $x \leq -1$.

Consider first the class Q . If an odd prime p occurs to an odd exponent in $v = F_{k+2}$, then Lemma 3.1(i) and the first two equations in the first row of (13) imply

$$p \mid ax + 1, \quad p \mid bx + 1.$$

Hence

$$p \mid b(ax + 1) - a(bx + 1) = b - a = r,$$

contrary to Lemma 3.1(i). Therefore every odd prime occurs to an even exponent in F_{k+2} , and so F_{k+2} is a square or twice a square, which is impossible. The same argument applies to $Q + C$, using the third row of (13).

For $Q + P$ and $Q + P + C$, use instead an odd prime occurring to an odd exponent in $\ell_2 = L_{k+2}$ and Lemma 3.1(ii). We conclude that every odd prime occurs to an even exponent in L_{k+2} . Since $k + 2$ is odd, formula

(12) shows that $v_2(L_{k+2})$ is also even. Thus L_{k+2} is a square, contradicting Cohn's theorem because $k+2 \geq 5$. $\square$

We can now prove the final part of the main theorem.

Proof of Theorem 1.1(iii). By Lemma 4.1, every integral point must belong to one of the eight old classes. As noted above, the elimination in [4, Theorem 4], together with its Fibonacci specialization in [4, Theorem 6], eliminates all of these except $P + 2E'_k(\mathbb{Q})$. By the standard 2-descent criterion [4, Proposition 1], an integral point (x, y) in this class satisfies

$$F_{2k}x + 1 = \square, \quad F_{2k+2}x + 1 = \square, \quad F_{2k+4}x + 1 = \square.$$

Since $c = F_{2k+4} > 1$, an integer satisfying these three square conditions cannot be negative. The Hoggatt-Bergum theorem [2], together with the trivial solution $x = 0$, therefore gives

$$x = 0, \quad x = 4F_{2k+1}F_{2k+2}F_{2k+3}.$$

Substitution gives precisely (2) and (3). $\square$

5. THE GENERIC RANKS OF THE TWO PARITY FAMILIES

The identity

$$L_k^2 - 5F_k^2 = 4(-1)^k$$

suggests separating the odd and even indices already at the level of one-parameter families. We now show that the resulting generic ranks are exactly 2 and 1, respectively.

Let T be an indeterminate. A parametrization of

$$\ell^2 - 5f^2 = -4$$

is

$$(14) \quad f = \frac{T^2 - 2T + 5}{T^2 - 5}, \quad \ell = -\frac{T^2 - 10T + 5}{T^2 - 5},$$

while a parametrization of

$$\ell^2 - 5f^2 = 4$$

is

$$(15) \quad f = -\frac{4T}{T^2 - 5}, \quad \ell = -\frac{2(T^2 + 5)}{T^2 - 5}.$$

In both cases put

$$u = \frac{\ell + f}{2}, \quad v = \frac{\ell + 3f}{2}, \quad \ell_1 = \frac{\ell + 5f}{2}, \quad \ell_2 = \frac{3\ell + 5f}{2},$$

and

$$a = f\ell, \quad b = u\ell_1, \quad c = v\ell_2.$$

We denote by $\mathcal{E}_-$ and $\mathcal{E}_+$ the curves

$$y^2 = (ax + 1)(bx + 1)(cx + 1)$$

over $\mathbb{Q}(T)$ obtained from (14) and (15), respectively.

For the odd family, the relation $\ell^2 - 5f^2 = -4$, with $\ell = 2u - f$, gives

$$u^2 - fu - f^2 = -1.$$

Hence the calculation in Proposition 2.1 is valid identically over $\mathbb{Q}(T)$, with F_{k-1} replaced by $u - f$ and F_{2k+1} by $f^2 + u^2$. Thus $\mathcal{E}_-$ has the two rational sections

$$\mathcal{P} = (0, 1), \quad \mathcal{Q} = \left(-\frac{u-f}{\ell uv}, \frac{f^2+u^2}{\ell uv} \right).$$

The even family has the rational section $\mathcal{P} = (0, 1)$.

We use the injective specialization criterion of Gusić and Tadić [9]. In the form needed here, let

$$\mathcal{E} : \quad V^2 = (U - e_1)(U - e_2)(U - e_3), \quad e_i \in \mathbb{Z}[T]$$

be a nonconstant elliptic curve over $\mathbb{Q}(T)$. Suppose that the specialization at $T = T_0 \in \mathbb{Q}$ is nonsingular and that every nonconstant square-free divisor in $\mathbb{Z}[T]$ of each of

$$(e_1 - e_2)(e_1 - e_3), \quad (e_2 - e_1)(e_2 - e_3), \quad (e_3 - e_1)(e_3 - e_2)$$

takes a non-square value in $\mathbb{Q}$. Then the specialization homomorphism at T_0 is injective.

Theorem 5.1. *The generic Mordell-Weil ranks of the two families are*

$$\text{rank } \mathcal{E}_-(\mathbb{Q}(T)) = 2, \quad \text{rank } \mathcal{E}_+(\mathbb{Q}(T)) = 1.$$

Proof. We first consider $\mathcal{E}_-$. Put $d = T^2 - 5$. Starting from $\mathcal{E}_\pm$, make the changes

$$X = abc x, \quad Y = abc y, \quad Z = X + ab.$$

Then

$$Y^2 = Z(Z + M)(Z + N), \quad M = a(c - b), \quad N = b(c - a).$$

Since M and N have denominators dividing d^4 , the further change

$$U = d^4 Z, \quad V = d^6 Y$$

gives an integral model

$$V^2 = U(U + m_\pm)(U + n_\pm), \quad m_\pm = d^4 M, \quad n_\pm = d^4 N.$$

Here $e_1 = 0$, $e_2 = -m_\pm$ and $e_3 = -n_\pm$ belong to $\mathbb{Z}[T]$, as required by the specialization criterion. Moreover,

$$j = 256 \frac{(m_\pm^2 - m_\pm n_\pm + n_\pm^2)^3}{m_\pm^2 n_\pm^2 (m_\pm - n_\pm)^2}.$$

The formulas below show directly that this is nonconstant (for example, $\deg m_- = 8$, $\deg n_- = 7$, whereas $\deg m_+ = 7$, $\deg n_+ = 8$).

For the odd family, direct simplification gives

$$\begin{aligned} m_- &= -(T^2 - 10T + 5)(T^2 - 2T + 5) \\ &\quad \times (T^4 + 4T^3 + 30T^2 + 20T + 25), \\ n_- &= 16T(T^2 + 5)(T^4 + 30T^2 + 25), \end{aligned}$$

and

$$\begin{aligned} n_- - m_- &= (T^2 + 2T + 5)(T^2 + 10T + 5) \\ &\quad \times (T^4 - 4T^3 + 30T^2 - 20T + 25). \end{aligned}$$

The displayed factors are irreducible over $\mathbb{Q}$. We take

$$T_0 = \frac{85}{38}.$$

This is the Fibonacci specialization corresponding to $k = 17$, since

$$\frac{L_{17} - 1}{F_{17} - 1} = \frac{85}{38}.$$

The absolute square classes in $\mathbb{Q}^*/\mathbb{Q}^{*2}$ of the three irreducible factor values of m_- are

$$5 \cdot 3571, \quad 5 \cdot 1597, \quad 73 \cdot 149 \cdot 2221,$$

those of the three factor values of n_- are

$$2 \cdot 5 \cdot 17 \cdot 19, \quad 3 \cdot 5 \cdot 107, \quad 7 \cdot 23 \cdot 103681,$$

and those of the three factor values of $n_- - m_-$ are

$$5 \cdot 37 \cdot 113, \quad 5 \cdot 9349, \quad 5 \cdot 13 \cdot 141961.$$

In each union of the irreducible-factor lists occurring in m_-n_- , $m_-(n_- - m_-)$, and $n_-(n_- - m_-)$, every nonempty product has a nontrivial absolute square class. Since n_- has content 16, the first and third products in condition (A) also admit the constant square-free factor 2; adjoining the square class of 2 still leaves all nonempty products nontrivial. Hence, for either choice of sign, every nonconstant square-free divisor in condition (A) of [9] specializes to a nonsquare in $\mathbb{Q}$. Therefore the specialization criterion applies and specialization at $T = 85/38$ is injective.

At this value, (14) gives

$$(f, \ell) = (1597, 3571) = (F_{17}, L_{17}),$$

so the specialization of $\mathcal{E}_-$ is E_{17} ; the integral model above specializes to a $\mathbb{Q}$ -isomorphic curve. It is known that

$$\text{rank } E_{17}(\mathbb{Q}) = 2$$

(see [4, Section 5]). On the other hand, the sections $\mathcal{P}$ and $\mathcal{Q}$ specialize at $k = 17$ to the independent points P_{17} and Q_{17} by Proposition 3.3. Thus

$$2 \leq \text{rank } \mathcal{E}_-(\mathbb{Q}(T)) \leq \text{rank } E_{17}(\mathbb{Q}) = 2.$$

For the even family the same construction gives

$$V^2 = U(U + m_+)(U + n_+),$$

where

$$\begin{aligned} m_+ &= 16T(T^2 + 5)(T^4 + 8T^3 + 30T^2 + 40T + 25), \\ n_+ &= (T^2 + 2T + 5)(T^2 + 10T + 5) \\ &\quad \times (3T^4 + 20T^3 + 90T^2 + 100T + 75), \end{aligned}$$

and

$$\begin{aligned} n_+ - m_+ &= (T + 1)(T + 5)(3T^2 + 10T + 15) \\ &\quad \times (T^4 + 4T^3 + 30T^2 + 20T + 25). \end{aligned}$$

Again the displayed factorizations are into irreducible polynomials over $\mathbb{Q}$. We take $T_0 = 20/9$. The square classes of the irreducible factor values are

$$\begin{array}{c|ccc} m_+ & 5 & 5 \cdot 7 \cdot 23 & 17 \cdot 53 \cdot 109 \\ n_+ & 5 \cdot 233 & 5 \cdot 521 & 3 \cdot 90481 \end{array}$$

and

$$n_+ - m_+ \mid 29 \quad 5 \cdot 13 \quad 3 \cdot 5 \cdot 281 \quad 3001.$$

Again, every nonempty product in each of the three relevant unions has a nontrivial absolute square class. Since m_+ has content 16, the first and second products in condition (A) also admit the constant square-free factor 2; including it creates no square relation. Therefore the same is true after either choice of sign, and condition (A) of [9] holds. Hence specialization is injective.

Formula (15) at $T = 20/9$ gives

$$(f, \ell) = (144, 322) = (F_{12}, L_{12}),$$

so the specialization of $\mathcal{E}_+$ is E_{12} , and the integral model specializes to a $\mathbb{Q}$ -isomorphic curve. The computation

$$\text{rank } E_{12}(\mathbb{Q}) = 1$$

is recorded in [4, Section 5]. The section $\mathcal{P}$ is non-torsion. Indeed, its specialization P_{12} is non-torsion by [4, Theorem 3 and Remark 4]. We conclude

$$1 \leq \text{rank } \mathcal{E}_+(\mathbb{Q}(T)) \leq \text{rank } E_{12}(\mathbb{Q}) = 1.$$

□

6. RANK DISTRIBUTION AND A HEURISTIC

The additional point Q_k gives a simple explanation for a phenomenon observed computationally in [5, Remark 4.9.5]. The even and odd subfamilies should be treated separately.

For even k , the family has the standard forced point P_k , so the minimal possible rank is 1. For odd $k \geq 3$, Theorem 1.1(ii) gives two independent forced points, so the minimal possible rank is 2.

Computations with Magma for $1 \leq k \leq 100$ gave

$$\#\{k : W(E_k) = +1\} = 52, \quad \#\{k : W(E_k) = -1\} = 48.$$

More strikingly, the same split occurs in each parity class: among the 50 odd values and among the 50 even values there are 26 curves with root number $+1$ and 24 with root number -1 .

Using complete factorizations of the relevant Fibonacci and Lucas numbers, we extended the root-number computation to $1 \leq k \leq 709$. The results are

	$W(E_k) = +1$	$W(E_k) = -1$
k even	162	192
k odd	193	162

and hence, in the full range,

$$\#\{k : W(E_k) = +1\} = 355, \quad \#\{k : W(E_k) = -1\} = 354.$$

Thus the root numbers are almost perfectly balanced overall, although the two parity subfamilies exhibit noticeable imbalances in opposite directions. This is consistent with the expected asymptotic equidistribution, while suggesting that convergence within the two parity subfamilies may be rather slow.

Combining the parity conjecture with the usual minimalist philosophy that the rank is normally the smallest one compatible with the forced points and the root number, one expects

	$W(E_k) = -1$	$W(E_k) = +1$
k even	1	2
k odd	3	2

If the root numbers are asymptotically equidistributed in each parity subfamily, this leads to the following heuristic.

Conjecture 6.1. As $k \rightarrow \infty$, the curves E_k have ranks 1, 2, 3 with densities

$$\frac{1}{4}, \quad \frac{1}{2}, \quad \frac{1}{4},$$

respectively, while the set of k for which $\text{rank } E_k(\mathbb{Q}) \geq 4$ has density 0. If, in addition, the contribution of these higher-rank specializations to the average is negligible, then the average rank should tend to 2.

The computations for $k \leq 100$ are remarkably consistent with this prediction. The PARI/GP calculations give

rank interval	[1, 1]	[2, 2]	[3, 3]	[1, 3]	[2, 4]	[3, 5]	[2, 6]
number	20	45	21	5	6	2	1.

If the actual rank equals the lower endpoint of every unresolved interval, the resulting counts are

$$25, \quad 52, \quad 23$$

for ranks 1, 2, 3, respectively. These are exactly the counts predicted from the computed root numbers and the two forced-rank baselines, with the exceptional value $k = 1$ taken into account.

For completeness, the 2-Selmer ranks

$$\dim_{\mathbf{F}_2} \text{Sel}_2(E_k) - 2$$

for $1 \leq k \leq 100$ were distributed as follows:

r_2	1	2	3	4	5	6	7
#	11	31	33	20	3	1	1.

These data show that the family also has interesting 2-Selmer phenomena, but the abundance of odd- k curves expected to have rank 3 is already naturally explained by the point Q_k , together with the parity heuristic.

Remark 6.2. Theorem 5.1 gives a geometric counterpart to the minimal-rank heuristic: the odd and even Fibonacci specializations lie on one-parameter families of generic ranks 2 and 1, respectively. Beyond the parity conjecture, the remaining heuristic inputs in Conjecture 6.1 are the expected equidistribution of root numbers, together with the usual expectation that rank jumps by at least two have density zero.

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