# GENERALIZED FIBONACCI NUMBERS AND THE PROBLEM OF DIOPHANTUS

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# 1 Introduction

Let $n$ be an integer. A set of positive integers $\{a_1, \ldots, a_m\}$ is said to have *the property of Diophantus of order* $n$, symbolically $D(n)$, if for all $i, j = 1, \ldots, m$, $i \neq j$, the following holds: $a_i a_j + n = b_{ij}^2$, where $b_{ij}$ is an integer. The set $\{a_1, \ldots, a_m\}$ is called *a Diophantine* $m$*-tuple.*

In this paper we construct several Diophantine quadruples whose elements are represented using generalized Fibonacci numbers. It is a generalization of the following statements (see [8], [12], [6]):

the sets

$$\{F_{2n}, F_{2n+2}, F_{2n+4}, 4F_{2n+1}F_{2n+2}F_{2n+3}\},$$

$$\{n, n+2, 4n+4, 4(n+1)(2n+1)(2n+3)\}$$

have the property $D(1)$, and the set

$$\{2F_{n-1}, 2F_{n+1}, 2F_n^3 F_{n+1}F_{n+2}, 2F_{n+1}F_{n+2}F_{n+3}(2F_{n+1}^2 - F_n^2)\}$$

has the property $D(F_n^2)$, for all positive integers $n$.

These results are applied to the Pell numbers and are used to obtain explicit formulas for quadruples with the property $D(l^2)$, where $l$ is an integer.

# 2 Preliminaries

## 2.1 The problem of Diophantus

The Greek mathematician Diophantus of Alexandria noted that the numbers $x$, $x+2$, $4x+4$ and $9x+6$, where $x = \frac{1}{16}$, have the following property: the product of any two of them increased by 1 is a square of a rational number ([3]). The French mathematician Pierre de Fermat first found a set with the property $D(1)$, and it was $\{1, 3, 8, 120\}$. Later, Davenport and Baker [2] showed that if there is a set $\{1, 3, 8, d\}$ with the property $D(1)$, then $d$ has to be 120.

In [5], the problem of the existence of Diophantine quadruples with the property $D(n)$ was considered for an arbitrary integer $n$. The following result was proved:

*If an integer $n$ is not of the form $4k+2$ and $n \notin \{3, 5, 8, 12, 20, -1, -3, -4\}$, then there exists a quadruple with the property $D(n)$.*

Non-existence of Diophantine quadruples with the property $D(4k+2)$ was proved in [1] and [5].

In [5], the sets with the property $D(l^2)$ were particularly discussed. It was proved that *for any integer $l$ and any set $\{a, b\}$ with the property $D(l^2)$, where $ab$ is not a perfect square, there exists an infinite number of sets of the form $\{a, b, c, d\}$ with the property $D(l^2)$.* We would like to describe the construction of those sets.

Let $ab + l^2 = k^2$ and let $s$ and $t$ be positive integers satisfying the Pellian equation $S^2 - abT^2 = 1$ ($s$ and $t$ exist since $ab$ is not a perfect square). Two double sequences $y_{n,m}$ and $z_{n,m}$, $n, m \in \mathbf{Z}$, can be defined as follows as in [5]:

$$
\begin{gathered}
y_{0,0} = l, \quad z_{0,0} = l, \quad y_{1,0} = k + a, \quad z_{1,0} = k + b, \\
y_{-1,0} = k - a, \quad z_{-1,0} = k - b, \\
y_{n+1,0} = \tfrac{2k}{l} y_{n,0} - y_{n-1,0}, \quad z_{n+1,0} = \tfrac{2k}{l} z_{n,0} - z_{n-1,0}, \quad n \in \mathbf{Z}, \\
y_{n,1} = s y_{n,0} + a t z_{n,0}, \quad z_{n,1} = b t y_{n,0} + s z_{n,0}, \quad n \in \mathbf{Z}, \\
y_{n,m+1} = 2s y_{n,m} - y_{n,m-1}, \quad z_{n,m+1} = 2s z_{n,m} - z_{n,m-1}, \quad n, m \in \mathbf{Z}.
\end{gathered}
$$

Let us write

$$x_{n,m} = (y_{n,m}^2 - l^2)/a. \tag{1}$$

According to [5, Theorem 2], if $x_{n,m}$ and $x_{n+1,m}$ are positive integers then the set $\{a, b, x_{n,m}, x_{n+1,m}\}$ has the property $D(l^2)$. It is also proved that the sets $\{a, b, x_{0,m}, x_{1,m}\}$, $m \in \mathbf{Z} \setminus \{-1, 0\}$, and $\{a, b, x_{-1,m}, x_{0,m}\}$, $m \in \mathbf{Z} \setminus \{0, 1\}$, have the property $D(l^2)$. So, it is sufficient to find one positive integer solution of the Pellian equation $S^2 - abT^2 = 1$ to extend a set $\{a, b\}$ with the property $D(l^2)$ to a set $\{a, b, c, d\}$ with the same property.

## 2.2 Generalized Fibonacci numbers

In [9] the Generalized Fibonacci sequence of numbers $(w_n)$ was defined by Horadam as follows: $w_n = w_n(a, b; p, q)$, $w_0 = a$, $w_1 = b$, $w_n = pw_{n-1} - qw_{n-2}$ $(n \geq 2)$, where $a, b, p, q$ are integers. The properties of that sequence were discussed in detail in [10], [11] and [13]. The following identities have been proved:

$$w_n w_{n+2r} - eq^n U_r = w_{n+r}^2 \tag{2}$$

$$4w_n w_{n+1}^2 w_{n+2} + (eq^n)^2 = (w_n w_{n+2} + w_{n+1}^2)^2 \tag{3}$$

$$w_n w_{n+1} w_{n+3} w_{n+4} = w_{n+2}^4 + eq^n(p^2 + q) w_{n+2}^2 + e^2 q^{2n+1} p^2 \tag{4}$$

$$4w_nw_{n+1}w_{n+2}w_{n+4}w_{n+5}w_{n+6}+e^2q^{2n}(w_nU_4U_5-w_{n+1}U_2U_6-w_nU_1U_8)^2$$
$$=(w_{n+1}w_{n+2}w_{n+6}+w_nw_{n+4}w_{n+5})^2\,. \tag{5}$$

Here $e=pab-qa^2-b^2$ and $U_n=w_n(0,1;p,q)$. The identity (5) is due to Morgado ([13]).

Our purpose is to apply the above identities at constructing Diophantine quadruples. Considering the construction described in 2.1, we will restrict our attention to two special cases. For simplicity of notation these are

$$u_n=u_n(p)=w_n(0,1;p,-1),\ \ p\geq 1,$$

$$g_n=g_n(p)=w_n(0,1;p,1),\ \ p\geq 2.$$

The Fibonacci sequence $F_n=u_n(1)$, the Pell sequence $P_n=u_n(2)$, the Fibonacci numbers of even subscript $F_{2n}=g_n(3)$, and $g_n(2)=n$ are important special cases of the above sequences.

Apart from the sequences $(u_n)$ and $(g_n)$, we also wish to investigate joined sequences $(v_n)$ and $(h_n)$, which are defined by $v_n=u_{n-1}+u_{n+1}$, $h_n=g_{n+1}-g_{n-1}$. It is easy to check that

$$v_n=w_n(2,p;p,-1)\,,$$

$$h_n=w_n(2,p;p,1)\,.$$

## 3 Quadruples with the property $D(p^2u_n^2)$ and $D(h_n^2)$

For every positive integer $n$,

$$4u_nu_{n+2}+(pu_{n+1})^2=v_{n+1}^2\,. \tag{6}$$

Indeed, $v_{n+1}^2-(pu_{n+1})^2=(u_n+u_{n+2})^2-(u_{n+2}-u_n)^2=4u_nu_{n+2}$. From the above, it follows that the sets $\{2u_n,2u_{n+2}\}$, $\{u_n,4u_{n+2}\}$, $\{4u_n,u_{n+2}\}$ have the property $D(p^2u_{n+1}^2)$. In order to extend these sets to the quadruples with the property $D(p^2u_{n+1}^2)$, by applying the construction described in 2.1, it is necessary to find a solution of the Pellian equation $S^2-4u_nu_{n+2}T^2=1$. One solution of this equation we can get from the identity

$$4u_nu_{n+1}^2u_{n+2}+1=(u_{n+1}^2+u_nu_{n+2})^2\,. \tag{7}$$

which is the direct consequence of (2). Therefore, we will set $s=u_{n+1}^2+u_nu_{n+2}$, $t=u_{n+1}$. Applying now the construction from 2.1 we get an infinite number of sets with the property $D(p^2u_{n+1}^2)$. Particularly, we have

**Theorem 1** *Let $n$ and $p$ be positive integers. Then the six sets*

$$\{2u_n,\ 2u_{n+2},\ 2p^2u_{n+1}^3(u_{n+1}-u_n)(u_{n+2}-u_n),\ 2p^2u_{n+1}^3(u_n+u_{n+1})(u_{n+1}+u_{n+2})\},$$

$$\{2u_n,\ 2u_{n+2},\ 2p^2u_{n+1}^3(u_n+u_{n+1})(u_{n+1}+u_{n+2}),$$
$$2(u_n+u_{n+1})(u_{n+1}+u_{n+2})(u_n+2u_{n+1}+u_{n+2})(u_nu_{n+1}+2u_nu_{n+2}+u_{n+1}u_{n+2})\},$$

$$\{u_n,\ 4u_{n+2},\ (u_{n+1}-u_n)(u_{n+2}-u_{n+1})(2u_{n+2}$$
$$-u_n-u_{n+1})(2u_{n+1}u_{n+2}-u_nu_{n+1}-u_nu_{n+2}),$$
$$p^2u_{n+1}^3(u_n+2u_{n+1})(u_{n+1}+2u_{n+2})\},$$

$$\{u_n,\ 4u_{n+2},\ p^2u_{n+1}^3(u_n+2u_{n+1})(u_{n+1}+2u_{n+2}),$$
$$(u_n+u_{n+1})(u_{n+1}+u_{n+2})(u_n+3u_{n+1}+2u_{n+2})(u_nu_{n+1}+3u_nu_{n+2}+2u_{n+1}u_{n+2})\},$$

$$\{4u_n,\ u_{n+2},\ (u_{n+1}-u_n)(u_{n+1}+u_{n+2}-2u_n)(u_nu_{n+2}+u_{n+1}u_{n+2}-2u_nu_{n+1}),$$
$$p^2u_{n+1}^3(2u_n+u_{n+1})(2u_{n+1}+u_{n+2})\},$$

$$\{4u_n,\ u_{n+2},\ p^2u_{n+1}^3(2u_n+u_{n+1})(2u_{n+1}+u_{n+2}),$$
$$(u_n+u_{n+1})(u_{n+1}+u_{n+2})(2u_n+3u_{n+1}+u_{n+2})(2u_nu_{n+1}+3u_nu_{n+2}+u_{n+1}u_{n+2})\}$$

*have the property* $D(p^2u_{n+1}^2)$.

**Proof:** The main idea of the proof is to show that the six sets in Theorem 1 are of the form $\{a,b,x_{0,1},x_{1,1}\}$ or $\{a,b,x_{-1,1},x_{0,1}\}$. Combining (6) with (7) we obtain $l=pu_{n+1}$, $k=v_{n+1}$, $s=u_{n+1}^2+u_nu_{n+2}$, $t=u_{n+1}$. To simplify notation, we write $u_{n+2}=A$, $u_{n+1}=B$. Hence, according to (2), $A^2-pAB-B^2=(-1)^{n+1}$ and that gives

$$(A^2-pAB-B^2)^2=1\,. \tag{8}$$

We arrange the proof in three parts, each part relating to two of the six sets.

**1°** $\underline{a=2u_n,\ b=2u_{n+2}}$

With notation in 2.1, we have

$$y_{0,0}=z_{0,0}=pu_{n+1},\ y_{1,0}=3u_n+u_{n+2},\ z_{1,0}=u_n+3u_{n+2},$$
$$y_{-1,0}=pu_{n+1},\ z_{-1,0}=-pu_{n+1}.$$

From this we obtain

$$\begin{aligned}
y_{0,1}&=pB[A^2+(2-p)AB-(2p-1)B^2]\\
y_{1,1}&=4A^3+(8-7p)A^2B+(3p^2-10p+4)AB^2+p(2p-3)B^3\\
y_{-1,1}&=pB[A^2-(p+2)AB+(2p+1)B^2]\,.
\end{aligned}$$

Relation (8) will be used to represent expressions of $x_{i,1}$, $i=-1,0,1$, obtained by putting $y_{i,1}$ in (1), as homogeneous polynomials in two variables $A$ and $B$. When those polynomials are factored, we have

$$\begin{aligned}
x_{0,1}&=2p^2B^3[A-(p-1)B](A+B)\\
&=2p^2u_{n+1}^3(u_n+u_{n+1})(u_{n+1}+u_{n+2})
\end{aligned}$$

$$
\begin{aligned}
x_{1,1} &= 2[A-(p-1)B](A+B)[2A-(p-2)B][2A^2-2(p-1)AB-pB^2] \\
&= 2(u_n+u_{n+1})(u_{n+1}+u_{n+2})(u_n+2u_{n+1}+u_{n+2})(u_nu_{n+1}+2u_nu_{n+2}+u_{n+1}u_{n+2}) \\
x_{-1,1} &= 2p^2B^3[(p+1)B-A](A-B) \\
&= 2p^2u_{n+1}^3(u_{n+1}-u_n)(u_{n+2}-u_{n+1})\,.
\end{aligned}
$$

**2°** $\underline{a=u_n,\ b=4u_{n+2}}$

We now have

$$
y_{0,0}=z_{0,0}=pu_{n+1},\ y_{1,0}=2u_n+u_{n+2},\ z_{1,0}=u_n+5u_{n+2},\\
y_{-1,0}=u_{n+2},\ z_{-1,0}=u_n-3u_{n+2}.
$$

Hence

$$
\begin{aligned}
y_{0,1} &= pB[A^2-(p-1)AB-(p-1)B^2] \\
y_{1,1} &= 3A^3-(5p-6)A^2B+(2p^2-7p+3)AB^2+p(p-2)B^3 \\
y_{-1,1} &= A^3-(p+2)A^2B+(p+1)AB^2+pB^3\,,
\end{aligned}
$$

and, from (1) and (8),

$$
\begin{aligned}
x_{0,1} &= p^2B^3(A+2B)[2A-(p-1)B] \\
&= p^2u_{n+1}^3(u_n+2u_{n+1})(u_{n+1}+2u_{n+2}) \\
x_{1,1} &= [A-(p-1)B](A+B)[3A-(p-3)B][3A^2-3(p-1)AB-pB^2] \\
&= (u_n+u_{n+1})(u_{n+1}+u_{n+2})(u_n+3u_{n+1}+2u_{n+2})(u_nu_{n+1}+3u_nu_{n+2}+2u_{n+1}u_{n+2}) \\
x_{-1,1} &= [A-(p-1)B][A-(p+1)B](A-B)[A^2-(p+1)AB-pB^2] \\
&= (2u_{n+2}-u_n-u_{n+1})(u_{n+1}-u_n)(u_{n+2}-u_{n+1})(2u_{n+1}u_{n+2}-u_nu_{n+1}-u_nu_{n+2})\,.
\end{aligned}
$$

**3°** $\underline{a=4u_n,\ b=u_{n+2}}$

In this case

$$
y_{0,0}=z_{0,0}=pu_{n+1},\ y_{1,0}=5u_n+u_{n+2},\ z_{1,0}=u_n+2u_{n+2},\\
y_{-1,0}=u_{n+2}-3u_n,\ z_{-1,0}=u_n.
$$

According to that

$$
\begin{aligned}
y_{0,1} &= pB[A^2-(p-4)AB-(4p-1)B^2] \\
y_{1,1} &= 6A^3-(11p-12)A^2B+(5p^2-16p+6)AB^2+p(4p-5)B^3 \\
y_{-1,1} &= -2A^3+(5p+4)A^2B-(3p^2+8p+2)AB^2+p(4p+3)B^3\,,
\end{aligned}
$$

and finally

$$
\begin{aligned}
x_{0,1} &= p^2B^3(A+2B)[2A-(2p-1)B] \\
&= p^2u_{n+1}^3(2u_{n+1}+u_{n+2})(2u_n+u_{n+1}) \\
x_{1,1} &= [A-(p-1)B](A+B)[3A-(2p-3)B][3A^2-3(p-1)AB-2pB^2] \\
&= (u_n+u_{n+1})(u_{n+1}+u_{n+2})(2u_n+3u_{n+1}+u_{n+2})(2u_nu_{n+1}+3u_nu_{n+2}+u_{n+1}u_{n+2}) \\
x_{-1,1} &= [A-(p+1)B][A-(2p+1)B](A-B)[A^2-(p+1)AB+2pB^2] \\
&= (u_{n+1}-u_n)(u_{n+2}-u_{n+1})(u_{n+1}+u_{n+2}-2u_n)(u_nu_{n+2}+u_{n+1}u_{n+2}-2u_nu_{n+1})\,.
\end{aligned}
$$

■

Using the identities

$$4g_ng_{n+2}+h_{n+1}^2=p^2g_{n+1}^2$$

$$4g_ng_{n+1}^2g_{n+2}+1=(g_{n+1}^2+g_ng_{n+2})^2$$

we find the following theorem may be proved in much the same way as Theorem 1.

**Theorem 2** *Let $m \geq 1$ and $p \geq 2$ be integers. Then the six sets*

$$\{2g_n,\, 2g_{n+2},\, 2g_{n+1}h_{n+1}^2(g_{n+1}-g_n)(g_{n+2}-g_{n+1}),\, 2g_{n+1}h_{n+1}^2(g_n+g_{n+1})(g_{n+1}+g_{n+2})\},$$

$$\{2g_n,\, 2g_{n+2},\, 2g_{n+1}h_{n+1}^2(g_n+g_{n+1})(g_{n+1}+g_{n+2}),\\ 2(p+2)g_{n+1}(g_n+g_{n+1})(g_{n+1}+g_{n+2})(g_ng_{n+1}+2g_ng_{n+2}+g_{n+1}g_{n+2})\},$$

$$\{g_n,\, 4g_{n+1},\, (g_{n+1}-g_n)(g_{n+2}-g_{n+1})(2g_{n+2}-g_n-g_{n+1})(2g_{n+1}g_{n+2}-g_ng_{n+1}-g_ng_{n+2}),\\ g_{n+1}h_{n+1}^2(g_n+2g_{n+1})(g_{n+1}+2g_{n+2})\},$$

$$\{g_n,\, 4g_{n+2},\, g_{n+1}h_{n+1}^2(g_n+2g_{n+1})(g_{n+1}+g_{n+2}),\\ (g_n+g_{n+1})(g_{n+1}+g_{n+2})(g_n+3g_{n+1}+2g_{n+2})(g_ng_{n+1}+3g_ng_{n+2}+2g_{n+1}g_{n+2})\},$$

$$\{4g_n,\, g_{n+2},\, (g_{n+1}-g_n)(g_{n+2}-g_{n+1})(g_{n+1}+g_{n+2}-2g_n)(g_ng_{n+2}+g_{n+1}g_{n+2}-2g_ng_{n+1}),\\ g_{n+1}h_{n+1}^2(2g_n+g_{n+1}+g_{n+2})\},$$

$$\{4g_n,\, g_{n+2},\, g_{n+1}h_{n+1}^2(2g_n+g_{n+1})(2g_{n+1}+g_{n+2}),\\ (g_n+g_{n+1})(g_{n+1}+g_{n+2})(2g_n+3g_{n+1}+g_{n+2})(2g_ng_{n+1}+3g_ng_{n+2}+g_{n+1}g_{n+2})\}$$

*have the property $D(h_{n+1}^2)$.*

## 4 Morgado identity

We are now going to use the Morgado identity (5). It is easy to check that

$$w_nU_4U_5-w_{n+1}U_2U_6-w_nU_1U_8=U_2U_3(w_{n+4}-qw_{n+2})\,,$$

$$w_{n+1}w_{n+2}w_{n+6}+w_nw_{n+4}w_{n+5}=w_{n+3}(eq^nU_2^2U_3+2w_{n+2}w_{n+4})\,.$$

If we restrict the discussion to the sequences $(u_n)$ and $(g_n)$, the Morgado identity can be used as a base for constructing quadruples with the property $D((u_2u_3v_{n+3})^2)$ and $D((g_2g_3h_{n+3})^2)$.

We are again going to use the construction described in 2.1. This time it is not necessary to use the solutions of the Pellian equation. We will try to choose the numbers $a$ and $b$ in the manner that the solution of the problem can be obtained using only the sequence $(x_{n,0})$. According to 2.1, if $x_{2,0} \in \mathbf{N}$ or $x_{-2,0} \in \mathbf{N}$ then respectively $\{a, b, x_{1,0}, x_{2,0}\}$ and $\{a, b, x_{-1,0}, x_{-2,0}\}$ are Diophantine quadruples.

Since $y_{2,0} = \frac{2k}{l}(k+a) - l$, $y_{-2,0} = \frac{2k}{l}(k-a) - l$, we have

$$x_{2,0} = \frac{y_{2,0}^2 - l^2}{a} = \frac{4k(k+a)(k+b)}{l^2} = \frac{4k}{l^2}(kx_{1,0} - l^2)\,,$$
$$x_{-2,0} = \frac{y_{-2,0}^2 - l^2}{a} = \frac{-4k(k-a)(k-b)}{l^2} = \frac{4k}{l^2}(kx_{-1,0} + l^2)\,.$$

**Theorem 3** *Let $n$ and $p$ be positive integers and* $k = u_{n+3}[2u_{n+2}u_{n+4} - (-1)^n p^2(p^2+1)]$. *Then the three sets*

$$\{2u_nu_{n+1}u_{n+2},\ 2u_{n+4}u_{n+5}u_{n+6},\ 2(p^2+1)^2u_{n+3}v_{n+3}^2,\ 4k(\tfrac{2ku_{n+3}}{p^2} - 1)\},$$
$$\{2u_nu_{n+1}u_{n+4},\ 2u_{n+2}u_{n+5}u_{n+6},\ 2p^2u_{n+3}v_{n+3}^2,\ 4k(\tfrac{2ku_{n+3}}{(p^2+1)^2} + 1)\},$$
$$\{2u_nu_{n+2}u_{n+5},\ 2u_{n+1}u_{n+4}u_{n+6},\ 2u_{n+3}v_{n+3}^2,\ 4k(\tfrac{2ku_{n+3}}{p^2(p^2+1)^2} - 1)\}$$

*have the property $D(p^2(p^2+1)^2v_{n+3}^2)$.*

**Proof:** The proof is by applying the construction from 2.1 to the relation (5) for $w_n = u_n$. Three cases need to be considered.

**1°** $\underline{a = 2u_nu_{n+1}u_{n+2},\ b = 2u_{n+4}u_{n+5}u_{n+6}}$

Hence, $a + b = 2(p^2+2)u_{n+3}[(p^2+1)(u_{n+2}^2 + u_{n+4}^2) + (p^2-1)u_{n+2}u_{n+4}]$. This gives

$$\begin{aligned} x_{1,0} &= a + b + 2k \\ &= 2(p^2+1)^2u_{n+3}(u_{n+2}+u_{n+4})^2 = 2(p^2+1)^2u_{n+3}v_{n+3}^2\,, \\ x_{2,0} &= 4k(\frac{k\cdot 2(p^2+1)^2u_{n+3}v_{n+3}^2}{p^2(p^2+1)^2v_{n+3}^2} - 1) = 4k(\frac{2ku_{n+3}}{p^2} - 1)\,. \end{aligned}$$

**2°** $\underline{a = 2u_nu_{n+1}u_{n+4},\ b = 2u_{n+2}u_{n+5}u_{n+6}}$

Now we have $a + b = 2u_{n+3}[(p^2+1)(p^2+4)u_{n+2}u_{n+4} - u_{n+2}^2 - u_{n+4}^2]$ and

$$\begin{aligned} x_{-1,0} &= a + b - 2k = 2p^2u_{n+3}v_{n+3}^2\,, \\ x_{-2,0} &= 4k(\frac{k\cdot 2p^2u_{n+3}v_{n+3}^2}{p^2(p^2+1)^2v_{n+3}^2} + 1) = 4k(\frac{2ku_{n+3}}{(p^2+1)^2} + 1)\,. \end{aligned}$$

**3°** $\underline{a = 2u_n u_{n+2} u_{n+5},\;\; b = 2u_{n+1} u_{n+4} u_{n+6}}$

We have $a + b = 2(p^2 + 2)u_{n+3}[u_{n+2}^2 + u_{n+4}^2 - (p^2 + 1)u_{n+2}u_{n+4}]$ and

$$x_{1,0} = 2u_{n+3}v_{n+3}^2\,,$$
$$x_{2,0} = 4k\Big(\frac{2ku_{n+3}}{p^2(p^2+1)^2} - 1\Big)\,.$$

It remains to prove that all elements of the sets from this theorem are integers. It is sufficient to prove that the number $\dfrac{8k^2u_{n+3}}{p^2(p^2+1)^2}$ is an integer for all positive integers $n$. That is the direct consequence of the relation

$$\frac{8k^2u_{n+3}}{p^2(p^2+1)^2} = \frac{8u_{n+3}^3[p^4(p^2+1)^2 - (-1)^n 4p^2(p^2+1)u_{n+2}u_{n+4} + 4u_{n+2}^2u_{n+4}^2]}{u_2^2u_3^2}$$

and the fact that $u_2|u_{2m}$ and $u_3|u_{3m}$ for all $m \in \mathbf{N}$, which is easy to prove by induction. ■

The following theorem can be proved in much the same way as Theorem 3.

**Theorem 4** *Let $n \geq 1$ and $p \geq 2$ be integers and* $k = g_{n+3}[2g_{n+2}g_{n+4} - p^2(p^2-1)]$. *Then the three sets*

$$\{2g_ng_{n+1}g_{n+2},\, 2g_{n+4}g_{n+5}g_{n+6},\, 2(p^2-1)^2g_{n+3}h_{n+3}^2,\, 4k(\tfrac{2kg_{n+3}}{p^2} + 1)\},$$
$$\{2g_ng_{n+1}g_{n+4},\, 2g_{n+2}g_{n+5}g_{n+6},\, 2p^2g_{n+3}h_{n+3}^2,\, 4k(\tfrac{2kg_{n+3}}{(p^2-1)^2} - 1)\},$$
$$\{2g_ng_{n+2}g_{n+5},\, 2g_{n+1}g_{n+4}g_{n+6},\, 2g_{n+3}h_{n+3}^2,\, 4k(\tfrac{2kg_{n+3}}{p^2(p^2-1)^2} + 1)\}$$

*have the property $D(p^2(p^2-1)^2h_{n+3}^2)$.*

We now want to show that the sequence $(g_n)$ possesses another interesting property based on the identity

$$g_ng_{n+1}g_{n+3}g_{n+4} + [(p \pm 1)g_{n+2}]^2 = (g_{n+2}^2 \pm p)^2\,. \tag{9}$$

Let us prove this relation. From (4) we have

$$g_ng_{n+1}g_{n+3}g_{n+4} = g_{n+2}^4 - (p^2+1)g_{n+2}^2 + p^2 = (g_{n+2}^2 \pm p)^2 - (p \pm 1)^2g_{n+2}^2\,.$$

Now, the construction described in 2.1 can be applied on the relation (9). We have $a = g_ng_{n+1},\; b = g_{n+3}g_{n+4},\; k = g_{n+2}^2 \pm p,$ which gives

$$\begin{aligned} x_{\mp 1,0} &= a + b \mp 2k = (p^3 - 3p \mp 2)g_{n+2}^2 = (p \pm 1)^2(p \mp 2)g_{n+2}^2\,, \\ x_{\mp 2,0} &= 4(g_{n+2}^2 \pm p)(g_{n+1} \mp g_n)(g_{n+4} \mp g_{n+3})\,. \end{aligned}$$

We have thus proved

**Theorem 5** *Let $n \geq 1$ and $p \geq 2$ be integers. Then the set*

$$\{g_n g_{n+1},\, g_{n+3}g_{n+4},\, (p+1)^2(p-2)g_{n+2}^2,\, 4(g_{n+2}^2+p)(g_{n+1}-g_n)(g_{n+4}-g_{n+3})\}$$

*has the property $D((p+1)^2 g_{n+2}^2)$ and the set*

$$\{g_n g_{n+1},\, g_{n+3}g_{n+4},\, (p-1)^2(p+2)g_{n+2}^2,\, 4(g_{n+2}^2-p)(g_{n+1}+g_n)(g_{n+3}+g_{n+4})\}$$

*has the property $D((p-1)^2 g_{n+2}^2)$.*

## 5 Generalization of a result of Bergum

Bergum and Hoggatt have proved (see [8]) that the set

$$\{F_{2n}, F_{2n+2}, F_{2n+4}, 4F_{2n+1}F_{2n+2}F_{2n+3}\} \tag{10}$$

has the property $D(1)$ for every positive integer $n$. It has been proved in [4] that the set

$$\{F_{2n}, F_{2n+4}, 5F_{2n+2}, 4L_{2n+1}F_{2n+2}L_{2n+3}\} \tag{11}$$

also has the property $D(1)$ and in [5] the quadruples with the property $D(4)$, $D(9)$ and $D(64)$ have been found using Fibonacci numbers. We now want to extend these results to the sequences $(u_n)$ and $(g_n)$ starting from the identity (2). Applying (2) to the sequence $(u_n)$ we get

$$u_{2n} \cdot u_{2n+2r} + u_r^2 = u_{2n+r}^2\,. \tag{12}$$

Therefore, the sets $\{u_{2n}, u_{2n+2}\}$ and $\{u_{2n}, u_{2n+4}\}$ have respectively the property $D(1)$ and $D(p^2)$, for every positive integer $n$. It was shown in the section 4 that if $a$, $b$, $k$ and $l$ are the positive integers such that $ab + l^2 = k^2$ and if the number $\pm 4k(k \pm a)(k \pm b)/l^2$ is a positive integer, then the set $\{a, b, a+b \pm 2k, \pm 4k(k \pm a)(k \pm b)/l^2\}$ has the property $D(l^2)$. According to this, we have

**Theorem 6** *Let $n$ and $p$ be positive integers. Then the sets*

$$\{u_{2n},\, u_{2n+2},\, 2u_{2n}+(p-2)u_{2n+1},\, 4u_{2n+1}[(p-2)u_{2n+1}^2+2u_{2n}u_{2n+1}+1]\},$$
$$\{u_{2n},\, u_{2n+2},\, 2u_{2n}-(p-2)u_{2n+1},\, 4u_{2n+1}[2u_{2n+1}u_{2n+2}-(p-2)u_{2n+1}^2-1]\}$$

*have the property $D(1)$ and the set*

$$\{u_{2n},\, u_{2n+4},\, p^2 u_{2n+2},\, 4u_{2n+1}u_{2n+2}u_{2n+3}\}$$

*has the property $D(p^2)$.*

For the sequence $(g_n)$ we can prove even a stronger result, namely from (2) we have

$$g_n \cdot g_{n+2r} + g_r^2 = g_{n+r}^2 \,. \tag{13}$$

for every (not only even) positive integer $n$. Starting now from the sets $\{g_n, g_{n+2}\}$ and $\{g_n, g_{n+4}\}$ with the property $D(1)$ and $D(p^2)$ respectively, we find the following result may be proved in much the same way as the Theorem 6.

**Theorem 7** *Let $n \geq 1$ and $p \geq 2$ be integers. Then the sets*

$$\{g_n,\, g_{n+2},\, (p-2)g_{n+1},\, 4g_{n+1}[(p-2)g_{n+1}^2 + 1]\},$$
$$\{g_n,\, g_{n+2},\, (p+2)g_{n+1},\, 4g_{n+1}[(p+2)g_{n+1}^2 - 1]\}$$

*have the property $D(1)$, and the set*

$$\{g_n,\, g_{n+4},\, p^2 g_{n+2},\, 4g_{n+1}g_{n+2}g_{n+3}\}$$

*has the property $D(p^2)$.*

**Example 1** It is easy to check that $g_n(3) = w_n(0,1;3,1) = F_{2n}$ (generally, $u_{2n}(p) = pg_n(p^2+2)$ ). We conclude from the Theorem 7, for $p = 3$, that the sets (10) and (11) have the property $D(1)$ and the set

$$\{F_{2n}, F_{2n+8}, 9F_{2n+4}, 4F_{2n+2}F_{2n+4}F_{2n+6}\}$$

has the property $D(9)$.

## 6 Application to the Pell numbers and polynomials

In this section, we apply the results discussed in the previous section to the some special cases of the sequences $(u_n)$ and $(g_n)$. The case of the Fibonacci sequence $F_n = u_n(1)$ and the joined Lucas sequence $L_n = v_n(1)$ is studied in detail in [6].

Let us first examine the Pell sequence $P_n = u_n(2)$ and the Pell-Lucas sequence $Q'_n = v_n(2)$. All elements of the sequence $(Q'_n)$ are even numbers, so we can put $Q'_n = 2Q_n$. The numbers $P_n$ and $Q_n$ are the solutions of the Pellian equation $x^2 - 2y^2 = \pm 1$. Namely, it is true that

$$Q_n^2 - 2P_n^2 = (-1)^n \,.$$

The sequences $(P_n)$ and $(Q_n)$ are related by relation $P_n + P_{n+1} = Q_{n+1}$. Applying this relation to the Theorem 1, we get:

**Corollary 1** *For every positive integer $n$, the sets*

$$\{P_n,\, P_{n+2},\, 4P_{n+1}^3 Q_n Q_{n+1},\, 4P_{n+1}^3 Q_{n+1} Q_{n+2}\},$$
$$\{P_n,\, P_{n+2},\, 4P_{n+1}^3 Q_{n+1} Q_{n+2},\, 4P_{n+2} Q_{n+1} Q_{n+2}[P_{n+1}P_{n+2} - (-1)^n]\}$$

*have the property $D(P_{n+1}^2)$.*

In [6], the quadruples with the property $D(L_{n+2}^2)$ are constructed using identities

$$4F_nF_{n+4} + L_{n+2}^2 = 9F_{n+2}^2 \tag{14}$$

$$4F_nF_{n+2}^2F_{n+4} + 1 = (F_{n+2}^2 + F_nF_{n+4})^2 \tag{15}$$

For the sequences $(u_n)$, the following analogues of above identities are valid:

$$4u_nu_{n+4} + (pv_{n+2})^2 = [(p^2+2)u_{n+2}]^2 \tag{16}$$

$$4u_nu_{n+2}^2u_{n+4} + p^4 = (u_{n+2}^2 + u_nu_{n+4})^2\,. \tag{17}$$

Unfortunately, existence of the term $p^4$ in (17) make it impossible to apply the construction for getting quadruples with the property $D(p^2v_{n+2}^2)$ from 2.1. But in the case $p=2$ the solution of the equation $S^2 - abT^2 = 4$ can be get from the relation (17). Therefore, we can apply the modified construction described in [5, Remark 1].

**Theorem 8** *For every positive integer $n$, the sets*

$$\{P_n,\ P_{n+4},\ 4P_{n+1}P_{n+2}P_{n+3}Q_{n+2}^2,\ 4P_{n+2}Q_{n+1}Q_{n+2}^2Q_{n+3}\},$$
$$\{P_n,\ P_{n+4},\ 4P_{n+2}Q_{n+1}Q_{n+2}^2Q_{n+3},\ 16P_{n+2}Q_{n+1}Q_{n+3}(2P_{n+2}^2 - P_{n+1}P_{n+3})\}$$

*have the property $D(4Q_{n+2}^2)$.*

**Proof:** The sets from the Theorem 8 are easily seen to be respectively of the form $\{a, b, x'_{-1,1}, x'_{0,1}\}$ and $\{a, b, x'_{0,1}, x'_{1,1}\}$, where the sequence $(x'_{n,m})$ is constructed as it is described in [5, Remark 1] by setting $a = P_n$, $b = P_{n+4}$, $s' = P_{n+2}^2 + P_nP_{n+4}$, $t' = P_{n+2}$. ■

In distinction from the identities (16) and (17), the construction from 2.1 can be directly applied to the following identities:

$$Q_nQ_{n+2} + Q_{n+1}^2 = 4P_{n+1}^2 \tag{18}$$

$$Q_nQ_{n+1}^2Q_{n+2} + 1 = 4P_{n+1}^4\,. \tag{19}$$

We have thus proved

**Theorem 9** *For every positive integer $n$, the sets*

$$\{Q_n,\ Q_{n+2},\ 4P_nP_{n+1}Q_{n+1}^3,\ 4P_{n+1}P_{n+2}Q_{n+1}^3\},$$
$$\{Q_n,\ Q_{n+2},\ 4P_{n+1}P_{n+2}Q_{n+1}^3,\ 4P_{n+1}P_{n+2}Q_{n+2}(P_{n+1}P_{n+3} - P_nP_{n+2})\}$$

*have the property $D(Q_{n+1}^2)$.*

Obviously, the Theorem 3 and 6 can also be applied to the sequence $(P_n)$. But applying Theorem 6, as it is done for Fibonacci numbers in [5, Theorem 3], gives more.

**Corollary 2** *For every positive integer $n$, the sets*

$$\{P_{2n},\, P_{2n+2},\, 2P_{2n},\, 4P_{2n+1}Q_{2n}Q_{2n+1}\},$$
$$\{P_{2n},\, P_{2n+2},\, 2P_{2n+2},\, 4P_{2n+1}Q_{2n+1}Q_{2n+2}\}$$

*have the property $D(1)$, the sets*

$$\{P_{2n},\, P_{2n+4},\, 4P_{2n+2},\, 4P_{2n+1}P_{2n+2}P_{2n+3}\},$$
$$\{P_{2n},\, P_{2n+4},\, 8P_{2n+2},\, 4P_{2n+2}Q_{2n+1}Q_{2n+3}\}$$

*have the property $D(4)$ and the set*

$$\{P_{2n},\, P_{2n+8},\, 36P_{2n+4},\, P_{2n+2}P_{2n+4}P_{2n+6}\}$$

*has the property $D(144)$.*

In this paper, only the quadruples with the property $D(n)$, where $n$ is a perfect square, have been examined. However, let us mention that the set $\{1, P_{2n+1}(3P_{2n+1}-2), 3P_{2n+1}^2-1, P_{2n+1}(3P_{2n+1}+2)\}$ has the property $D(-Q_{2n+1}^2)$, for every positive integer $n$.

Since $g_n(2) = n$, the results from this paper can be used to get the sets with the property of Diophantus whose elements are polynomials. For example, from Theorem 7 we get the Jones result that the set $\{n,\, n+2,\, 4(n+1),\, 4(n+1)(2n+1)(2n+3)\}$ has the property $D(1)$ for every positive integer $n$ (see [12]).

The following interesting property of the binomial coefficients can be obtained as a consequence of the results from the section 4:

for every positive integer $n \geq 4$, the sets

$$\{\binom{n-1}{3},\, \binom{n+3}{3},\, 6n,\, \frac{2n(n^2-7)(n^2-3n+1)(n^2+3n-1)}{3}\},$$

$$\{\binom{n-1}{3},\, \binom{n+3}{3},\, \frac{2n(n^2+2)}{3},\, \frac{2n(n^2-7)(n^3-3n^2+2n-3)(n^3+3n^2+2n+3)}{27}\}$$

have the property $D(1)$. Note that $h_n(2) = 2$.

Finally, let us mention that using these results the explicit formulas for quadruples with the property $D(l^2)$, for given integer $l$, can be obtained. Of course, only the sets with at least one element which is not divisible by $l$ are of any interest.

**Corollary 3** *Let $l$ be an integer. The sets*

$$\{(l-1)(l-2),\,(l+1)(l+2),\,4l^2,\,2(2l-3)(2l+3)(l^2-2)\},\ \textit{for } l \geq 3 \tag{20}$$

$$\{1,\,l^4-3l^2+1,\,l^2(l^2-1),\,4l^2(l^2-1)(l^2-2)\},\ \textit{for } l \geq 2 \tag{21}$$

*have the property $D(l^2)$.*

**Proof:** We can get the set (20) by putting $p=2$ and $n+2=l$ in the second set of Theorem 5.

Since $g_1(p)=1,\ g_3(p)=p^2-1,\ g_5(p)=p^4-3p^2+1,$ the set (21) can be obtained by putting $n=1$ and $p=l$ in the third set of Theorem 7. ∎

**Remark 1** One question still unanswered is whether any of the Diophantine quadruples from this paper can be extended to the Diophantine quintuple with the same property. In this connection, let us mention it is proved in [7] that for every integer $l$ and every set $\{a,b,c,d\}$ with the property $D(l^2)$, where $abcd \neq l^4$, there exists rational number $r$, $r \neq 0$, such that the set $\{a,b,c,d,r\}$ has the property that the product of any two its elements increased by $l^2$ is a square of a rational number.

For example, if the method from [7] is applied to the second set in the Corollary 3, we get

$$r = \frac{8l(l-1)(l+1)(l^2-2)(2l^2-3)(2l^4-4l^2+1)(2l^4-6l^2+3)}{[4(l-1)^2(l+1)^2(l^2-2)(l^2-l-1)(l^2+l-1)-1]^2}\,.$$

From this, for $l=2$, we have the set $\{89760,\,128881,\,644405,\,1546572,\,12372576\}$ with the property $D(4\cdot 359^4)$.

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