SQUARE ROOTS WITH MANY GOOD APPROXIMANTS

Andrej Dujella[1]
Department of Mathematics, University of Zagreb, Bijenička cesta 30, 10000 Zagreb, Croatia
`duje@math.hr`

Vinko Petričević
Dinarski put 1i, 10000 Zagreb, Croatia
`vinko@vip.hr`

Abstract

Let d be a positive integer that is not a perfect square. It was proved by Mikusiński in 1954 that if the period $s(d)$ of the continued fraction expansion of $\sqrt{d}$ satisfies $s(d) \leq 2$, then all Newton's approximants $R_n = \frac{1}{2}(\frac{p_n}{q_n} + \frac{dq_n}{p_n})$ are convergents of $\sqrt{d}$. If R_n is a convergent of $\sqrt{d}$, then we say that R_n is a good approximant. Let $b(d)$ denote the number of good approximants among the numbers R_n, $n = 0, 1, \ldots, s(d) - 1$. In this paper we show that the quantity $b(d)$ can be arbitrary large. Moreover, we construct families of examples which show that for every positive integer b there exist a positive integer d such that $b(d) = b$ and $b(d) > s(d)/2$.

1 Introduction

Let d be a positive integer that is not a perfect square. Then the simple continued fraction expansion of $\sqrt{d}$ has the form

$$\sqrt{d} = [a_0; \overline{a_1, a_2, \ldots, a_{s-1}, 2a_0}].$$

Here $s = s(d)$ denotes the length of the shortest period in the expansion of $\sqrt{d}$. Moreover, the sequence $a_1, \ldots, a_{s-1}$ is palindromic, i.e. $a_i = a_{s-i}$ for $i = 1, \ldots, s-1$. The expansion can be obtained by the following algorithm:

$$\begin{gathered} a_0 = \lfloor \sqrt{d} \rfloor, \quad s_1 = a_0, \quad t_1 = d - a_0^2, \\ a_{n-1} = \left\lfloor \frac{s_{n-1}+a_0}{t_{n-1}} \right\rfloor, \quad s_n = a_{n-1}t_{n-1} - s_{n-1}, \quad t_n = \frac{d-s_n^2}{t_{n-1}} \quad \text{for } n \geq 2 \end{gathered} \tag{1.1}$$

(see e.g. [15, p. 319]).

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Let $\frac{p_n}{q_n}$ denote the nth convergent of $\sqrt{d}$. Then

$$\frac{1}{(a_{n+1}+2)q_n^2} < \left|\sqrt{d} - \frac{p_n}{q_n}\right| < \frac{1}{a_{n+1}q_n^2} \tag{1.2}$$

(see e.g. [14, p. 23]). In particular, $|\sqrt{d} - \frac{p_n}{q_n}| < \frac{1}{q_n^2}$. Furthermore, by Legendre's theorem (see [14, Theorem 5C, p. 18]), if a rational number $\frac{p}{q}$ with $q \geq 1$ satisfies

$$\left|\sqrt{d} - \frac{p}{q}\right| < \frac{1}{2q^2}\,, \tag{1.3}$$

then $\frac{p}{q}$ is a convergent of $\sqrt{d}$.

Continued fractions provide one method for obtaining "good" rational approximations to $\sqrt{d}$. Another method for the approximation is by Newton's iterative method for solving nonlinear equations. Applying this method to the equation $f(x) = x^2 - d = 0$, we obtain the Newton's formula

$$x_{k+1} = \frac{1}{2}\left(x_k + \frac{d}{x_k}\right)\,. \tag{1.4}$$

We are interested in connections between these two methods of approximation. The main question is: if we assume that x_0 is a convergent of $\sqrt{d}$, is x_1 also a convergent of $\sqrt{d}$, i.e. if $x_0 = \frac{p_n}{q_n}$, we are asking whether

$$R_n := \frac{1}{2}\left(\frac{p_n}{q_n} + \frac{dq_n}{p_n}\right)$$

is also a convergent of $\sqrt{d}$?

This question has been discussed by several authors. It is well known (see e.g. [2, p. 468]) that

$$R_{ks-1} = \frac{p_{2ks-1}}{q_{2ks-1}}, \quad \text{for } k \geq 1. \tag{1.5}$$

It was proved by Mikusiński [11] that if $s = 2t$, then

$$R_{kt-1} = \frac{p_{2kt-1}}{q_{2kt-1}}, \quad \text{for } k \geq 1. \tag{1.6}$$

These results imply that if $s(d) \leq 2$, then all approximants R_n are convergents of $\sqrt{d}$. In 2001, Dujella [3] proved the converse of this result. Namely, if all approximants R_n are convergents of $\sqrt{d}$, then $s(d) \leq 2$.

Thus, if $s(d) > 2$, we know that some of the approximants R_n are convergents and some of them are not convergents. So we may ask how often we can obtain convergents. This question will be discussed in this paper.

2 Good approximants

The properties of continued fractions listed in the introduction (formulas (1.2) and (1.3)) will give us necessary and sufficient conditions for R_n to be a convergent. The conditions involve the greatest common divisor of the numerator and denominator of $R_n = \frac{p_n^2+dq_n^2}{2p_nq_n}$. Thus, in the next lemma we give some useful information about this quantity.

Lemma 1 *Let* $g := \gcd(p_n^2 + dq_n^2, 2p_nq_n)$. *Then* g *divides* $\gcd(2d, t_{n+1}, 2s_{n+1}, 2s_{n+2})$.

Proof. Since $\gcd(p_n, q_n) = 1$, we have that g divides $2p_n$ and $2d$. Now, the formulas

$$p_n^2 - dq_n^2 = (-1)^{n+1} t_{n+1}$$

and

$$p_np_{n-1} - dq_nq_{n-1} = (-1)^n s_{n+1}$$

(see e.g. [13, p. 92] and [4, Lemma 1]) imply that g divides also t_{n+1}, $2s_{n+1}$ and $2s_{n+2}$. ■

Now we obtain the following result, which is an improvement of [3, Proposition 2].

Proposition 1 *(i) If* $a_{n+1} > \frac{2}{g}\sqrt{\sqrt{d}+1}$, *then* R_n *is a convergent of* $\sqrt{d}$.

(ii) Assume that $a_i \neq 2$ *for all* $i \geq 1$. *If* $a_{n+1} > \frac{1}{g}\sqrt{3(\sqrt{d}+1)}$, *then* R_n *is a convergent of* $\sqrt{d}$.

Proof. (i) Let $R_n = \frac{u}{v}$, $\gcd(u, v) = 1$. Then $v = 2p_nq_n/g$. From [3, Lemma 2.1] and (1.2) we have

$$\left|R_n - \sqrt{d}\right| = \frac{q_n}{2p_n}\left(\frac{p_n}{q_n} - \sqrt{d}\right)^2 <$$

$$\frac{1}{2p_nq_n^3a_{n+1}^2} = \frac{2}{v^2g^2} \cdot \frac{p_n}{q_na_{n+1}^2} < \frac{1}{2v^2} \cdot \frac{4}{g^2a_{n+1}^2} \cdot (\sqrt{d}+1) < \frac{1}{2v^2},$$

which proves part (i) of the proposition.

(ii) We use a result of Koksma [8, p. 102] which says that if $a_i \neq 2$ for all $i \geq 1$ and

$$\left|\sqrt{d} - \frac{p}{q}\right| < \frac{1}{\frac{3}{2}q^2},$$

then p/q is a convergent of $\sqrt{d}$. Using the result just stated, the proof is completely analogous to the proof of part (i). ■

Proposition 2 *If $a_{n+1} < \frac{1}{g}\sqrt{2(\sqrt{d}-1)}-2$, then R_n is not a convergent of $\sqrt{d}$.*

Proof. We have

$$\left|R_n - \sqrt{d}\right| >$$

$$\frac{1}{2p_nq_n^3(a_{n+1}+2)^2} = \frac{2}{v^2g^2}\cdot\frac{p_n}{q_n(a_{n+1}+2)^2} > \frac{1}{v^2}\cdot\frac{2}{g^2(a_{n+1}+2)^2}\cdot(\sqrt{d}-1) > \frac{1}{v^2},$$

which proves the proposition. ∎

If R_n is a convergent of $\sqrt{d}$, then we say that R_n is a *good approximant.* Let

$$b(d) = |\{n : 0 \le n \le s(d)-1 \text{ and } R_n \text{ is a good approximant}\}|.$$

By [3, Theorem 3.2], if $s(d) > 2$ then $b(d) < s(d)$ (in fact, by [3, Lemma 2.4], $b(d) \le s(d)-2$). Komatsu [9] proved that if $d = (2n+1)^2+4$ then $b(d) = 3$, $s(d) = 5$ (see also [5]) and if $d = (2n+3)^2-4$ then $b(d) = 4$, $s(d) = 6$, while Dujella [3] proved that if $d = 16n^4-16n^3-12n^2+16n-4$, where $n \ge 2$, then $s(d) = 8$ and $b(d) = 6$.

Let

$$s_b = \min\{s : \text{there exists } d \text{ such that } s(d) = s \text{ and } b(d) = b\}.$$

Only five exact values of s_b are known: $s_1 = 1$, $s_2 = 2$, $s_3 = 5$, $s_4 = 6$ and $s_6 = 8$. In Table 1 we list upper bounds for s_b obtained by experiments with $d < 2\cdot 10^9$. (The bold values indicate precise values instead of upper bounds.) This table extends [3, Table 2], which - like this one - was also obtained by experiments. These tables give raise to the following questions (which the first author already asked in [3]).

Questions:

1) Is it true that $\inf\{s_b/b : b \ge 3\} = \frac{4}{3}$?

2) What can be said about $\sup\{s_b/b : b \ge 1\}$?

Trivially, we have

$$1 \le \inf\{\frac{s_b}{b} : b \ge 3\} \le \frac{4}{3},$$

since $s_b > b$ for $b \ge 3$ and there is a b with $s_b/b = 4/3$, namely $b = 6$. For the second question we have, by considering $b = 3$, that

$$\frac{5}{3} \le \sup\{\frac{s_b}{b} : b \ge 1\}.$$

In the next section, we will present some results concerning the second question. Our theoretical results will significantly improve some of the entries in Table 1. This will be done by considering sequences of d's which are given by exponential functions in n instead of polynomials in n as above or expressions obtained from the Fibonacci sequence as in [3]. The conditions from Propositions 1 and 2 will enable us to get our results.

b	$s_b \leq$	d	$s_b/b \leq$	b	$s_b \leq$	d	$s_b/b \leq$
3	5	13	**1.66667**	27	75	398641237	2.77778
4	6	21	**1.50000**	28	56	227136	2.00000
5	9	1450	1.80000	29	87	1978205	3.00000
6	8	108	**1.33334**	30	58	88452	1.93333
7	11	36125	1.57143	31	99	1381250	3.19354
8	12	76	1.50000	32	68	1946880	2.12500
9	17	280865	1.88889	33	127	49691210	3.84848
10	14	500	1.40000	34	78	76208384	2.29412
11	23	123370	2.09091	35	129	48946825	3.68571
12	18	141456	1.50000	36	80	1332144	2.22222
13	27	166634	2.07692	37	137	479833250	3.70270
14	22	5800	1.57143	38	92	8472240	2.42105
15	39	74356325	2.60000	39	133	929305	3.41026
16	22	94382820	1.37500	40	90	184548	2.25000
17	43	308125	2.52941	41	155	1724645	3.78049
18	32	52272	1.77778	42	98	690034333	2.33333
19	41	60125	2.15789	43	151	406445	3.51163
20	32	3201660	1.60000	44	112	35010157	2.54545
21	41	21125	1.95238	45	175	6331625	3.88889
22	40	2151864	1.81818	46	106	5491827	2.30435
23	65	97674013	2.82609	47	155	5415605	3.29788
24	38	53508	1.58333	48	104	1383840	2.16667
25	69	253045	2.76000	49	195	269131250	3.97959
26	50	29403	1.92308	50	124	5410080	2.48000

Table 1: upper bounds for s_b

3 Sequences with many good approximants

Our first aim is to prove that the quantity $b(d)$ can be arbitrary large, i.e.

$$\sup\{b(d) : d \text{ is a positive non-square integer}\} = +\infty.$$

Moreover, we would like to derive good general estimates for s_b/b. If we want $b(d)$ to be large, then we need that $s(d)$ is large. In the papers of Hendy [7], Bernstein [1], Williams [16, 17], Levesque [10], Halter-Koch [6] and Mollin [12] (among others), one can find many examples of families of positive integers d with large $s(d)$. More precisely, in these examples d is an exponential functions in an integer parameter n, while $s(d)$ is a linear function in n. E.g., in [16], it was proved that for

$$d = (q(4qk-1)^n - k)^2 + (4qk-1)^n \tag{3.1}$$

it holds that $s(d) = 3n+1$.

According to Proposition 1, we are particularly interested in those examples in which there are many large partial quotients a_i.

Proposition 3 *If $d_n = 3^{2n} - 3^n + 1$ for $n \geq 1$, then $s(d_n) = 3n+1$ and $b(d_n) = n+1$.*

Proof. Inserting $q = k = 1$ in (3.1), we obtain $d = d_n$. Therefore, the above mentioned result from [16] implies that $s(d_n) = 3n+1$. Alternatively, we can insert $l = q = c = \tau = \mu = -\lambda = 1$ in the main result of [6]. In both papers, we can also find information on partial quotients a_i and quantities s_i, t_i from the algorithm (1.1). We have $a_0 = 3^n - 1$,

$$\begin{aligned}
s_{3k+1} &= 3^n - 1, \quad t_{3k+1} = 3^{n-k}, \quad a_{3k+1} = 2\cdot 3^k - 1 \\
s_{3k+2} &= 3^n - 3^{n-k} + 1, \quad t_{3k+2} = 2\cdot 3^n - 3^{n-k} - 3^{k+1} + 2, \quad a_{3k+2} = 1 \\
s_{3k+3} &= 3^n - 3^{k+1} + 1, \quad t_{3k+3} = 3^{k+1}, \quad a_{3k+3} = 2\cdot 3^{n-k-1} - 1,
\end{aligned}$$

for $k = 0, 1, \ldots, n-1$.

By direct computation, we can check the statement of the proposition for $n = 1, 2, 3$. Therefore, we may assume that $n \geq 4$.

Let us first consider approximants of the form R_{3k}. From $g \,|\, 2d_n$ and $g \,|\, t_{n+1}$ we find that $g = 1$. We may apply Proposition 1 (ii), and we obtain that R_{3k} is a good approximant if

$$2\cdot 3^k - 1 > \sqrt{3(\sqrt{d}+1)}. \tag{3.2}$$

For $k > 1$ we have $2\cdot 3^k - 1 \geq \frac{17}{9}\cdot 3^k$ and $\sqrt{3(\sqrt{d}+1)} < \sqrt{3(3^n+1)} < \frac{7}{4}\cdot 3^{n/2} < \frac{17}{9}\cdot 3^{n/2}$. Thus, condition (3.2) is clearly satisfied for $k \geq \frac{n}{2}$. Applying Proposition 2, we find that

R_{3k} is not a good approximant if

$$2 \cdot 3^k + 1 < \sqrt{2(\sqrt{d} - 1)}. \tag{3.3}$$

This implies that R_0 and R_3 are not good approximants, and we may assume that $k \geq 2$. Since $2 \cdot 3^k + 1 \leq 3^{k+0.7}$ and $\sqrt{2(\sqrt{d}-1)} > \sqrt{2 \cdot 3^n - 3} > 3^{n/2+0.3}$, we conclude that R_{3k} is not a good approximant if $k \leq \frac{n-1}{2}$. Hence, if $n = 2l$, then good approximants are exactly those R_{3k} for which $k = l, l+1, \dots, 2l$, and if $n = 2l+1$, then good approximants are exactly those R_{3k} for which $k = l+1, l+2, \dots, 2l+1$.

By [3, Lemma 2.4], the approximant R_{3k+2} is good if and only if the approximant $R_{s-(3k+2)-2} = R_{3(n-k-1)}$ is good. From what we have already proved, it follows that if $n = 2l$, then good approximants are exactly those R_{3k+2} for which $k = 0, 1, \dots, l-1$, and if $n = 2l+1$, then good approximants are exactly those R_{3k+2} for which $k = 0, 1, \dots, l-1$.

Finally, let us consider approximants of the form R_{3k+1}. If $n = 2k+1$, then by the general result of Mikusiński (1.6), we have that $R_{3k+1} = R_{s/2-1}$ is a good approximant. Assume that $n \neq 2k+1$. For $g = \gcd(p_{3k+1}^2 + d_n q_{3k+1}^2, 2p_{3k+1}q_{3k+1})$, by Lemma 1, we have $g \mid 2(3^n - 3^{n-k} + 1)$ and $g \mid 2(3^n - 3^{k+1} + 1)$. Let us assume that $k < \frac{n-1}{2}$. The case $k \geq \frac{n}{2}$ can be treated in the same way (or we may apply [3, Lemma 2.4]). We obtain that g divides $2(3^{n-2k-1} - 1)$, and by our assumption, this number is not zero. Also, $g \mid 2(3^{2k+1} - 3^{k+1} + 1)$ and hence $4 \nmid g$. Therefore, $g \leq 3^{n-2k-1} - 1$ and if n is odd, then $g \leq \frac{1}{4}(3^{n-2k-1} - 1)$. On the other hand, if R_{3k+1} is a good approximant, since $a_{3k+2} = 1$ we get by Proposition 2 that $g \geq \frac{1}{3}\sqrt{2(\sqrt{d}-1)} - 2 > 3^{n/2-0.7}$. But, $3^{n-2k-1} - 1 > 3^{n/2-0.7}$ implies $n - 4k > 1$ (if n is odd, we obtain $n - 4k > 3$), while $2(3^{2k+1} - 3^{k+1} + 1) > 3^{n/2-0.7}$ implies $n - 4k \leq 4$. Therefore, the only possibilities are $n = 4k+2$ and $n = 4k+4$. Assume that $n = 4k+2$. Now we have that g divides $2(3^{2k+1} - 1)$ and $2(3^{2k+1} - 3^{k+1} + 1)$. For $k = 1$ we obtain $g \leq 2$, while for $k \geq 2$ we have $g \leq 2(3^{k+1} - 2) < 6 \cdot 3^{(n-2)/4} < 3^{n/4+1.14} \leq 3^{n/2-0.7}$, a contradiction. Assume now that $n = 4k+4$. Then g divides $2(3^{2k+3} - 1)$ and $2(3^{2k+3} - 3^{k+3} + 9)$. For $k \leq 2$ we obtain $g \leq 2$, while for $k \geq 3$ we have $g \leq 2(3^{k+3} - 10) < 3^{n/4+2.64} \leq 3^{n/2-0.7}$, a contradiction.

Putting these three cases together, we conclude that for $n = 2l$ the number of good approximants is $(l+1) + l + 0 = 2l + 1 = n + 1$, and for $n = 2l+1$ this number is $(l+1) + l + 1 = 2l + 2 = n + 1$. Thus, we proved that $b(d_n) = n + 1$. ■

Proposition 3 shows that

$$\sup\{b(d) : d \text{ is a positive non-square integer}\} = +\infty.$$

Moreover, it implies that

$$\sup\{\frac{s_b}{b} : b \geq 1\} \leq 3.$$

Now, we will improve the last result. We were not able to do it by considering a single sequence, so we will consider two sequences corresponding to even and odd b's, respectively.

First we handle the case of even b's.

Proposition 4 *If $d_n = (12 \cdot 9^n + 1)^2 + 6 \cdot 9^n$ for $n \geq 1$, then $s(d_n) = 4n + 6$.*

Proof. We claim that $a_0 = 12 \cdot 9^n + 1$,

$$\begin{aligned} s_{2k} &= 12 \cdot 9^n - 1, \quad t_{2k} = 9^k, \quad a_{2k} = 24 \cdot 9^{n-k}, \quad \text{for } k = 1, 2, \ldots, n, \\ s_{2k+1} &= 12 \cdot 9^n + 1, \quad t_{2k+1} = 6 \cdot 9^{n-k}, \quad a_{2k+1} = 4 \cdot 9^k, \quad \text{for } k = 0, 1, \ldots, n. \end{aligned}$$

Since $(12 \cdot 9^n + 2)^2 > d_n$, we have $a_0 = \lfloor \sqrt{d_n} \rfloor = 12 \cdot 9^n + 1$. The algorithm (1.1) gives

$$s_1 = 12 \cdot 9^n + 1, \quad t_1 = 6 \cdot 9^n, \quad a_1 = 4.$$

Now we will prove our claim by induction. We have checked that the claim is valid for $k = 0$. Assume that it is valid for $0, 1, 2, \ldots, k-1$, where $k \leq n$. Then

$$\begin{aligned} s_{2k} &= a_{2k-1} t_{2k-1} - s_{2k-1} = (4 \cdot 9^{k-1})(6 \cdot 9^{n-k+1}) - (12 \cdot 9^n + 1) = 12 \cdot 9^n - 1, \\ t_{2k} &= \frac{d_n - s_{2k-1}^2}{t_{2k-1}} = \frac{54 \cdot 9^n}{6 \cdot 9^{n-k+1}} = 9^k, \\ a_{2k} &= \left\lfloor \frac{s_{2k} + a_0}{t_{2k}} \right\rfloor = \frac{24 \cdot 9^n}{9^k} = 24 \cdot 9^{n-k}, \end{aligned}$$

and

$$\begin{aligned} s_{2k+1} &= (24 \cdot 9^{n-k}) \cdot 9^k - (12 \cdot 9^n - 1) = 12 \cdot 9^n + 1, \\ t_{2k+1} &= \frac{6 \cdot 9^n}{9^k} = 6 \cdot 9^{n-k}, \\ a_{2k+1} &= \left\lfloor \frac{24 \cdot 9^n + 2}{6 \cdot 9^{n-k}} \right\rfloor = 4 \cdot 9^k, \end{aligned}$$

which completes the proof of our claim.

Furthermore, we have

$$s_{2n+2} = 12 \cdot 9^n - 1, \quad t_{2n+2} = 9^{n+1}, \quad a_{2n+2} = \left\lfloor \frac{24 \cdot 9^n}{9^{n+1}} \right\rfloor = 2,$$

$$\begin{aligned} s_{2n+3} &= 2 \cdot 9^{n+1} - (12 \cdot 9^n - 1) = 6 \cdot 9^n + 1, \\ t_{2n+3} &= \frac{d_n - s_{2n+3}^2}{t_{2n+2}} = \frac{18 \cdot 9^n (6 \cdot 9^n + 1)}{9^{n+1}} = 2(6 \cdot 9^n + 1), \\ a_{2n+3} &= \left\lfloor \frac{18 \cdot 9^n + 2}{12 \cdot 9^n + 2} \right\rfloor = 1, \end{aligned}$$

$$
\begin{aligned}
s_{2n+4} &= 2(6\cdot 9^n+1)-(6\cdot 9^n+1)=6\cdot 9^n+1,\\
t_{2n+4} &= \frac{18\cdot 9^n(6\cdot 9^n+1)}{2(6\cdot 9^n+1)}=9^{n+1},\\
a_{2n+4} &= \left\lfloor \frac{18\cdot 9^n+2}{9^{n+1}} \right\rfloor = 2.
\end{aligned}
$$

We see that $s_{2n+3}=s_{2n+4}$ and, by [13, Chapter 24, Satz 3.10], it holds $s(d_n)=2(2n+3)=4n+6$. ■

Remark 1 Proposition 4 can be considered as a special case of the general result of Williams [17]. In [17], numbers of the form $d=(\sigma(qra^m+\mu(a^k+\lambda)/q)/2)^2-\sigma^2\mu\lambda a^m r$, with $\mu,\lambda=\{-1,1\}$, $qr\,|\,a^k+l$, $\gcd(m,k)=1$, $m>k\geq 1$, and $\sigma=1$ if $2\,|\,rqa^m+\mu(a^k+\lambda)/q$, while $\sigma=2$ otherwise, were studied. For $\mu=1$, $\lambda=-1$, $r=2$, $q=4$, $a=3$, $k=2$ and $\sigma=1$ we get $d=(4\cdot 3^m+1)^2+2\cdot 3^m$ and, since $\gcd(m,k)=1$, m has to be odd. From the general result on the periods of numbers of such form, it follows that $s(d)=2m+4$. For $m=2n+1$, we obtain $s(d_n)=2n+6$. However, since in the main result of [17] there are many cases to be considered and complete proofs are not given of each of them, we prefer to include the complete proof of Proposition 4 in our paper.

Next we calculate also $b(d_n)$ for the sequence d_n defined in the previous proposition, but before we can do so we need another lemma.

Lemma 2 *Let $d_n=(12\cdot 9^n+1)^2+6\cdot 9^n$ and $g_k=\gcd(p_k^2+d_nq_k^2,2p_kq_k)$. Then $g_{2l}=2$ and $g_{2l+1}=1$ for $l=0,1,\ldots,n$.*

Proof. By Lemma 1, we have that $g_k\,|\,\gcd(2d_n,t_{k+1},2s_{k+1},2s_{k+2})$. Since d_n is odd, g_k is not divisible by 4. Furthermore, $a_0=12\cdot 9^n+1$, $p_0=a_0$ and $q_0=1$ are odd. Since all a_i, $i=1,\ldots 2n+2$ are even, we conclude that all p_i are odd, while q_i is odd for even i, and q_i is even for odd i.

For $k=2l$ the quantity $p_k^2+d_nq_k^2$ is even and therefore $2\,|\,g_k$. Moreover, we have $g_k\,|\,(2s_{k+1}-2s_{k+2})=4$, which implies that $g_k=2$.

For $k=2l+1$, $l<n$, we also have $g_k\,|\,4$, and since in this case g_k is odd, we conclude that $g_k=1$.

For $k=2n+1$, we have $g_k\,|\,(4s_{2n+3}-2s_{2n+2})=6$. It is clear that g_k is odd and not divisible by 3. Thus, $g_k=1$. ■

Proposition 5 *Let $d_n=(12\cdot 9^n+1)^2+6\cdot 9^n$. Then $b(d_n)=2n+4$.*

Proof. By (1.5) and (1.6), we know that R_{2n+2} and R_{4n+5} are good approximants. By [3, Lemma 2.4], it suffices to check the approximants $R_0,R_1,\ldots R_{2n+1}$. From Propositions

1 and 2, it follows that R_k is a good approximant if $a_{k+1} \geq \frac{2\sqrt{12\cdot 9^n+2}}{g_k}$, while R_k is not good if $a_{k+1} \leq \frac{\sqrt{24\cdot 9^n}}{g_k} - 2$.

Consider first the case $k = 2l$, $l = 0, 1, \ldots, n$. Then $g_k = 2$, and R_k is a good approximant if

$$4 \cdot 9^l \geq 2\sqrt{3 \cdot 9^n + \frac{1}{2}}.$$

We have $2 \cdot 9^l > 3^{2l+0.6}$ and $\sqrt{3 \cdot 9^n + \frac{1}{2}} < \sqrt{3^{2n+1+0.2}} = 3^{n+0.6}$. It follows that R_{2l} is a good approximant if $l \geq \frac{n}{2}$. On the other hand, R_k is not a good approximant if

$$4 \cdot 9^l \leq \sqrt{6} \cdot 3^n - 2.$$

Since $3^{n+\frac{1}{2}}\sqrt{\frac{1}{2}} - 1 > 3^{n-0.3}$, we get the condition $3^{2l+0.7} \leq 3^{n-0.3}$, which implies that R_{2l} is not a good approximant if $l \leq \frac{n-1}{2}$. Hence, the number of good approximants in this case is $\lfloor \frac{n}{2} \rfloor + 1$.

Let us consider now the case $k = 2l - 1$, $l = 1, \ldots, n$. Now we have $g_k = 1$ and, accordingly, R_k is a good approximant if

$$24 \cdot 9^{n-l} \geq 4\sqrt{3 \cdot 9^n + \frac{1}{2}}$$

Since $6 \cdot 9^{n-l} > 3^{2n-2l+1+0.6}$ and $\sqrt{3 \cdot 9^n + \frac{1}{2}} < 3^{n+0.6}$ we obtain the condition $3^{2n-2l+1.6} \geq 3^{n+0.6}$, which implies that R_{2l-1} is a good approximant if $l \leq \frac{n+1}{2}$. Similarly, we have that R_k is not a good approximant if

$$24 \cdot 9^{n-l} \leq 2\sqrt{6} \cdot 3^n - 2.$$

From $24 \cdot 9^{n-l} < 4 \cdot 3^{2n-2l+1+0.7}$ and $3^n\sqrt{\frac{3}{2}} - \frac{1}{2} > 3^{n-0.3}$, we conclude that R_{2l-1} is not a good approximant if $l \geq \frac{n+2}{2}$. Hence, the number of good approximants in this case is $\lfloor \frac{n+1}{2} \rfloor$.

Finally, from $g_{2n+1} = 1$ and $a_{2n+2} = 2$ we see that R_{2n+1} is not a good approximant.

Therefore, among the approximants $R_0, R_1, \ldots, R_{2n+1}$ there are exactly $\lfloor \frac{n}{2} \rfloor + 1 + \lfloor \frac{n+1}{2} \rfloor = n + 1$ good approximants. Then, by [3, Lemma 2.4], we have also $n + 1$ good approximants among $R_{2n+3}, R_{2n+4}, \ldots, R_{4n+4}$. Taking into account that R_{2n+2} and R_{4n+5} are good approximants, we find that the total number of good approximants is $2n + 4$. ■

Propositions 4 and 5 together give the following corollary.

Corollary 1 *For $d_n = (12 \cdot 9^n + 1)^2 + 6 \cdot 9^n$ it holds $s(d_n) = 4n + 6$, $b(d_n) = 2n + 4$. Therefore, for every even positive integer b there exist a non-square positive integer d such that $b(d) = b$ and $b(d) > s(d)/2$.*

Next we study the case for odd b and to this extent we consider the sequence $d_n = (2 \cdot 9^n + 1)^2 + 9^n$.

Lemma 3 *Let $d_n = (2 \cdot 9^n + 1)^2 + 9^n$. Then $s(d_n) = 2n + 1$. Furthermore, it holds*

$$\begin{aligned} a_0 &= 2 \cdot 9^n + 1, \\ s_{2k} &= 2 \cdot 9^n - 1, \quad t_{2k} = 9^k, \quad a_{2k} = 4 \cdot 9^{n-k} \quad \textit{for } k = 1, 2, \ldots, n, \\ s_{2k+1} &= 2 \cdot 9^n + 1, \quad t_{2k+1} = 9^{n-k}, \quad a_{2k+1} = 4 \cdot 9^k \quad \textit{for } k = 0, 1, \ldots, n-1. \end{aligned}$$

Proof. See [7, Section 4]. ■

Lemma 4 *Let $d_n = (2 \cdot 9^n + 1)^2 + 9^n$. Then $g_k = \gcd(p_k^2 + d_n q_k^2, 2p_k q_k) = 1$.*

Proof. From $g_k \,|\, 2s_{k+1}$ and $g_k \,|\, 2s_{k+2}$ it follows that $g_k \,|\, 4$, while from $g_k \,|\, t_{k+1}$ we have that g_k is odd. Hence, $g_k = 1$. ■

Proposition 6 *Let $d_n = (2 \cdot 9^{2n} + 1)^2 + 9^{2n}$. Then $b(d_n) = 2n + 1$.*

Proof. By Proposition 1, R_k is a good approximant if $a_{k+1} \geq \frac{2\sqrt{2 \cdot 9^{2n}+2}}{g_k}$. We have $2\sqrt{2 \cdot 9^{2n} + 2} < 2\sqrt{2 \cdot 9^{2n+0.1}} < 2\sqrt{3^{4n+0.2+0.7}} = 2 \cdot 3^{2n+0.45}$.

By Proposition 2, the approximant R_k is not good if $a_{k+1} \leq \frac{\sqrt{2 \cdot 2 \cdot 9^{2n}}}{g_k} - 2$. We have $\sqrt{2 \cdot 2 \cdot 9^{2n}} - 2 = 2 \cdot (3^{2n} - 1) > 2 \cdot 3^{2n-0.2}$.

Assume now that $k = 2l$, $l = 0, 1, \ldots, 2n - 1$. Then $a_{2l+1}/2 = 2 \cdot 9^l \geq 3^{2n+0.45}$ and we obtain the following condition for good approximants: $3^{2l+0.6} \geq 3^{2n+0.45}$. Therefore, R_{2l} is a good approximant if $l \geq n$.

Since $a_{2l+1} = 4 \cdot 9^l \leq 2 \cdot 3^{2n-0.2}$, it follows that if $3^{2l+0.7} \leq 3^{2n-0.2}$, i.e. if $l \leq n - 1$, then R_{2l} is not a good approximant. Hence, the number of good approximants in this case is n.

By [3, Lemma 2.4], the approximant R_k is good if and only if the approximant R_{s-k-2} is good. Since the period $s(d_n)$ is odd, this fact implies that the number of good approximants among the numbers R_{2l+1}, $l = 0, 1, \ldots, 2n-1$ is also equal to n. Finally, by (1.5), we know that $R_{s-1} = R_{4n}$ is a good approximant.

Thus, we proved that among the numbers $R_0, R_1, \ldots, R_{4n}$ there are exactly $2n + 1$ good approximants. ■

By Lemma 4 and Proposition 6 we get the following:

Corollary 2 *For* $d_n = (2 \cdot 9^{2n} + 1)^2 + 9^{2n}$ *it holds* $s(d_n) = 4n + 1$ *and* $b(d_n) = 2n + 1$. *Therefore, for every odd positive integer* b *there exist a non-square positive integer* d *such that* $b(d) = b$ *and* $b(d) > s(d)/2$.

From Corollaries 1 and 2, we immediately obtain the following result.

Corollary 3

$$\sup\{\frac{s_b}{b} : b \geq 1\} \leq 2.$$

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