# Solving a family of quartic Thue inequalities using continued fractions

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**Abstract**

In this paper we find all primitive solutions of the Thue inequality

$$|x^4 + 2(1-n^2)x^2y^2 + y^4| \leq 2n+3,$$

where $n \geq 0$ is an integer.

## 1 Introduction

In 1909, Thue [15] proved that an equation $F(x,y) = \mu$, where $F \in \mathbb{Z}[X,Y]$ is a homogeneous irreducible polynomial of degree $n \geq 3$ and $\mu \neq 0$ a fixed integer, has only finitely many solutions. Such equations are called Thue equations. In 1968, Baker [1] gave an effective upper bound for the solutions of a Thue equation, based on his theory of linear forms in logarithms of algebraic numbers.

The first infinite parametrized family of Thue equations were considered by Thue himself in [16]. It was the family $(a+1)x^n - ay^n = 1$. In 1990, Thomas [14] investigated for the first time a parametrized family of cubic Thue equations of positive discriminant. Since then, several families of cubic, quartic and sextic Thue equations have been studied (see [8, 9] for references).

In [17], Tzanakis considered Thue equations of the form $F(x,y) = \mu$, where $F$ is a quartic form which corresponding quartic field $\mathbb{K}$ is the compositum of two real quadratic fields. Tzanakis showed that solving the equation $F(x,y) = \mu$, reduces to solving a system of Pellian equations. The

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Tzanakis method has been applied to several parametric families of quartic Thue equations and inequalities (see [5, 6, 7, 10, 11, 20]).

The application of Tzanakis' method for solving Thue equations of the special type has several advantages (see [17, 5]). Moreover, the additional advantages appear when we deal with Thue inequalities of the same type. Namely, the theory of continued fractions can be used in order to determine small values of $\mu$ for which the equation $F(x,y)=\mu$ has a solution. We can use classical results of Legendre and Fatou concerning Diophantine approximations of the form $|\alpha-\frac{a}{b}|<\frac{1}{2b^2}$ and $|\alpha-\frac{a}{b}|<\frac{1}{b^2}$, or their generalizations to the approximations of the form $|\alpha-\frac{a}{b}|<\frac{c}{b^2}$ for a positive real number $c$, due to Worley, Dujella and Ibrahimpašić (see [19, 3, 4]).

In present paper we will show another advantage of the application of Tzanakis method and continued fractions in solving parametric families of quartic Thue equations. Namely, in all mentioned results of that type, the authors were able to solve completely the corresponding system(s) of Pellian equations, and from these solutions it is straightforward to find all solutions of the Thue equation (inequality). However, even if the system of Pellian equations has nontrivial solutions (for some values of parameter) or the system cannot be solved completely, the information on solutions obtained from the theory of continued fractions and Diophantine approximations might be sufficient to show that the Thue equation has no solutions or has only trivial solutions. Note that the system and the original Thue equation are not equivalent: each solution of the Thue equation induces a solution of the system, but not vice-versa.

As an illustration of these phenomena we consider the family of Thue inequalities

$$|x^4+2(1-n^2)x^2y^2+y^4|\leq 2n+3, \tag{1}$$

for an integer $n\geq 3$.

In [11, 12], the third author considered the two-parametric family of Thue equations

$$x^4-2mnx^3y+2(m^2-n^2+1)x^2y^2+2mnxy^3+y^4=1 \tag{2}$$

and showed that, using Tzanakis method, it leads to solving the system of Pellian equations

$$V^2-(m^2+2)U^2=-2, \quad Z^2-(n^2-2)U^2=2.$$

She proved that if $m$ and $n$ are large enough and have sufficiently large common divisor, the all solutions of the system are $(U,V,Z)=(\pm1,\pm m,\pm n)$, which implies that Thue equation (2) has only trivial solutions $(x, y) = (\pm 1, 0)$, $(0, \pm 1)$. Note that inserting $m = 0$ in (2) we obtain

$$x^4 + 2(1 - n^2)x^2y^2 + y^4 = 1, \tag{3}$$

which is a special case of (1).

Our main result is

**Theorem 1** *Let $n \geq 3$ be an integer. If $2(n^2 - 1)$ is not a perfect square, then all primitive solutions of Thue inequality (1) are $(x, y) = (0, \pm 1)$, $(\pm 1, 0)$. If $2(n^2 - 1) = \nu^2$, then all primitive solutions of (1) are $(x, y) = (0, \pm 1)$, $(\pm 1, 0)$, $(\pm 1, \pm \nu)$, $(\pm \nu, \pm 1)$.*

We may complete our results by giving the information of the inequality (1) for $n = 0, 1, 2$. For $n = 0$ we have $(x^2 + y^2)^2 \leq 3$ which obviously has only trivial solutions $(x, y) = (0, \pm 1)$, $(\pm 1, 0)$, while for $n = 1$ we have $x^4 + y^4 \leq 5$, which has solutions $(x, y) = (0, \pm 1)$, $(\pm 1, 0)$, $(\pm 1, \pm 1)$. For $n = 2$ we have $|x^4 - 6x^2y^2 + y^4| = |(x^2 - 2xy - y^2)(x^2 + 2xy - y^2)| \leq 7$, for which it is easy to verify that all solutions are given by $(x, y) = (0, \pm 1)$, $(\pm 1, 0)$, $(\pm 1, \pm 1)$, $(\pm 1, \pm 2)$, $(\pm 2, \pm 1)$.

## 2 The system of Pellian equations

An application of the Tzanakis method to inequality (1), i.e. to the Thue equations

$$x^4 + 2(1 - n^2)x^2y^2 + y^4 = m \tag{4}$$

for $|m| \leq 2n + 3$, leads to the system of Pellian equations

$$\begin{aligned} U^2 - 2V^2 &= m, & (5) \\ Z^2 - (n^2 - 2)U^2 &= 2m, & (6) \end{aligned}$$

where $U = x^2 + y^2$, $V = nxy$, $Z = |n(x^2 - y^2)|$.

From (6) we have

$$\begin{aligned} \left| \sqrt{n^2 - 2} - \frac{Z}{U} \right| &= \left| (n^2 - 2) - \frac{Z^2}{U^2} \right| \cdot \left| \sqrt{n^2 - 2} + \frac{Z}{U} \right|^{-1} \\ &< \frac{2|m|}{U^2\sqrt{n^2 - 2}} \leq \frac{4n + 6}{\sqrt{n^2 - 2}} U^{-2} < \begin{cases} \frac{5}{U^2}, & n \geq 7 \\ \frac{6}{U^2}, & n = 4, 5, 6 \\ \frac{7}{U^2}, & n = 3. \end{cases} \end{aligned}$$

Now we can use Worley's extension [19, Theorem 1] of Legendre's theorem. Worley's result was slightly improved by Dujella in [3, Theorem 1].

**Theorem 2 (Worley, Dujella)** *Let $\alpha$ be a real number and let $a$ and $b$ be coprime nonzero integers, satisfying the inequality*

$$\left|\alpha - \frac{a}{b}\right| < \frac{c}{b^2},$$

*where $c$ is a positive real number. Let $\frac{p_k}{q_k}$ denotes the $k$th convergent in the continued fraction expansion of $\alpha$. Then $(a,b) = (rp_k \pm sp_{k-1}, rq_k \pm sq_{k-1})$, for some nonnegative integers $k$, $r$ and $s$ such that $rs < 2c$.*

By Theorem 2, we conclude that there exist nonnegative integers $k, r, s$, $rs < 14$, such that $Z = rp_k \pm sp_{k-1}$, $U = rq_k \pm sq_{k-1}$, where $\frac{p_k}{q_k}$ denotes the $k$th convergent in the continued fraction expansion of $\sqrt{n^2-2}$ (see [4] for more restrictive conditions on $r$ and $s$, and an explicit list of all possible pairs $(r,s)$).

The continued fraction expansion of $\sqrt{n^2-2}$ is periodic with the length of period equal to 4:

$$\sqrt{n^2-2} = [n-1, \overline{1, n-2, 1, 2n-2}].$$

In order to determine the values of $m$ for which equation (6) has a solution, we use the following result (see [6, Lemma 1]):

**Lemma 1** *Let $\alpha\beta$ be a positive integer which is not a perfect square, and let $p_k/q_k$ denotes the $k$th convergent of the continued fraction expansion of $\sqrt{\frac{\alpha}{\beta}}$. Let the sequences $(\sigma_k)$ and $(\tau_k)$ be defined by $\sigma_0 = 0$, $\tau_0 = \beta$ and*

$$a_k = \left\lfloor \tfrac{\sigma_k + \sqrt{\alpha\beta}}{\tau_k} \right\rfloor, \quad \sigma_{k+1} = a_k\tau_k - \sigma_k, \quad \tau_{k+1} = \tfrac{\alpha\beta - \sigma_{k+1}^2}{\tau_k} \quad \textit{for } k \geq 0.$$

*Then*

$$\alpha(rq_{k+1} + sq_k)^2 - \beta(rp_{k+1} + sp_k)^2 = (-1)^k(s^2\tau_{k+1} + 2rs\sigma_{k+2} - r^2\tau_{k+2}).$$

By Lemma 1, we obtain the following formula

$$(rp_k \pm sp_{k-1})^2 - (n^2-2)(rq_k \pm sq_{k-1})^2 = (-1)^k(s^2\tau_k \pm 2rs\sigma_{k+1} - r^2\tau_{k+1}), \quad (7)$$

where $(\sigma_0, \tau_0) = (0,1)$, $(\sigma_1, \tau_1) = (n-1, 2n-3)$, $(\sigma_2, \tau_2) = (n-2, 2)$, $(\sigma_3, \tau_3) = (n-2, 2n-3)$, $(\sigma_4, \tau_4) = (n-1, 1)$, $(\sigma_{k+4}, \tau_{k+4}) = (\sigma_k, \tau_k)$ for $k \geq 1$.

Inserting $k = 0, 1, 2, 3$ and all possibilities for $r$ and $s$ in (7), we obtain the following result.

**Proposition 1** *Let $m$ be an integer such that $|m| \leq 2n+3$ and such that the equation (6) has a solution in relatively prime integers $U$ and $Z$. Then $m \in M = \{1, -2n+3, 2n+3\}$ if $n \geq 7$, $m \in M \cup \{-4n+9\}$ if $n = 4, 5, 6$, and $m \in M \cup \{-6n+11\}$ if $n = 3$.*

*Furthermore, for $n \geq 3$, all solutions of this equation in relatively prime positive integers are given by $(U, Z) = (q_{4k+1}, p_{4k+1})$ if $m = 1$; $(U, Z) = (q_{4k} - q_{4k-1}, p_{4k} - p_{4k-1})$ and $(U, Z) = (q_{4k+3} + q_{4k+2}, p_{4k+3} + p_{4k+2})$ if $m = -2n+3$; $(U, Z) = (q_{4k} + 3q_{4k-1}, p_{4k} + 3p_{4k-1})$ and $(U, Z) = (3q_{4k+3} - q_{4k+2}, 3p_{4k+3} - p_{4k+2})$ if $m = 2n+3$, where $k \geq 0$ and $p_k/q_k$ denotes the $k$th convergent of continued fraction expansion of $\sqrt{n^2-2}$ and $(p_{-1}, q_{-1}) = (1, 0)$.*

*For $n = 4, 5, 6$ and additional $m = -4n+9$, all solutions of the equation (6) are given by $(U, Z) = (q_{4k} - 3q_{4k-1}, p_{4k} - 3p_{4k-1})$ and $(U, Z) = (3q_{4k+3} + q_{4k+2}, 3p_{4k+3} + p_{4k+2})$. For $n = 3$ and additional $m = -6n+11 = -7$, all solutions of the equation (6) are given by $(U, Z) = (q_{4k+1} + 2q_{4k}, p_{4k+1} + 2p_{4k})$.*

Therefore, we will study the equation (4) and the corresponding system (5)–(6) for $m = 1$, $-2n+3$ and $2n+3$.

First note that in the case $m = 1$, by a result of Walsh [18] (see also [12]) the equation (3) has only trivial solutions $(x, y) = (0, \pm 1)$, $(\pm 1, 0)$, unless $2(n^2-1)$ is not a perfect square. If $2(n^2-1) = \nu^2$, then all solutions of (3) are $(x, y) = (0, \pm 1)$, $(\pm 1, 0)$, $(\pm 1, \pm \nu)$, $(\pm \nu, \pm 1)$.

## 3 Case $m = -2n+3$

By Proposition 1, all solutions of the equation

$$Z^2 - (n^2-2)U^2 = 2(-2n+3) \tag{8}$$

in relatively prime nonnegative integers $U$ and $Z$ are given by $(U, Z) = (U_k, Z_k)$ or $(U, Z) = (U'_k, Z'_k)$, where

$$U_0 = 1, \quad U_1 = 2n^2 - 2n - 1, \quad U_k = 2(n^2-1)U_{k-1} - U_{k-2},$$

$$Z_0 = n - 2, \quad Z_1 = 2n^3 - 2n^2 - 3n + 2, \quad Z_k = 2(n^2-1)Z_{k-1} - Z_{k-2},$$

$$U'_0 = 1, \quad U'_1 = 2n - 1, \quad U'_k = 2(n^2-1)U'_{k-1} - U'_{k-2},$$

$$Z'_0 = -n + 2, \quad Z'_1 = 2n^2 - n - 2, \quad Z'_k = 2(n^2-1)Z'_{k-1} - Z'_{k-2},$$

Let us consider the other equation

$$U^2 - 2V^2 = -2n + 3. \tag{9}$$

It should be noted that the system (8)–(9) has solutions for some values of the parameter $n$. Namely, if $n-1$ is a perfect square, say $n=\nu^2+1$, then $U=1$, $V=\nu$, $Z=\nu^2-1$ is a solution. Also, if $(n-1)(2n+1)$ is a perfect square, then the system has a solution. E.g. for $n=433$ we have a solution $U=865$, $V=612$, $Z=374543$.

However, we are able to prove that the corresponding Thue equation

$$x^4+2(1-n^2)x^2y^2+y^4=-2n+3 \tag{10}$$

has no solutions for an integer $n\geq 3$. Indeed, assume that (10) has a solution $(x,y)$. Than the system (8)–(9) has a solution $(U,V,Z)$ such that $U=x^2+y^2$, $V=nxy$, $Z=|n(x^2-y^2)|$. In particular, $V\equiv 0 \pmod{n}$. On the other hand, it follows easily by induction that $U_k\equiv U'_k\equiv(-1)^k \pmod{n}$. Hence, $U^2\equiv 1 \pmod{n}$ and (9) implies $2V^2\equiv -2 \pmod{n}$, a contradiction.

## 4 Case $m=2n+3$

By Proposition 1, all solutions $U$ of the equation

$$Z^2-(n^2-2)U^2=2(2n+3) \tag{11}$$

in relatively prime nonnegative integers $U$ and $Z$ are given by $U=U_k$ or $U=U'_k$, where

$$U_0=1,\quad U_1=2n^2+2n-1,\quad U_k=2(n^2-1)U_{k-1}-U_{k-2},$$

$$U'_0=-1,\quad U'_1=2n+1,\quad U'_k=2(n^2-1)U'_{k-1}-U'_{k-2}.$$

Consider the other equation

$$U^2-2V^2=2n+3. \tag{12}$$

Again, the system (11)–(12) might have solutions for some values of the parameter $n$. E.g. if $(n+1)(2n-1)$ is a perfect square, as for $n=74$.

But, we prove that the corresponding Thue equation

$$x^4+2(1-n^2)x^2y^2+y^4=2n+3 \tag{13}$$

has no solutions for an integer $n\geq 3$. Indeed, assume that (13) has a solution $(x,y)$. That the system (11)–(12) has a solution $(U,V,Z)$ such that $U=x^2+y^2$, $V=nxy$, $Z=|n(x^2-y^2)|$. In particular, $V\equiv 0 \pmod{n}$. On the other hand, it follows easily by induction that $U_k\equiv(-1)^k \pmod{n}$ and $U'_k\equiv(-1)^{k+1} \pmod{n}$. Hence, $U^2\equiv 1 \pmod{n}$ and (12) implies $2V^2\equiv -2 \pmod{n}$, a contradiction.

## 5 Case $\gcd(U, Z) > 1$

We are working under assumption that $\gcd(x, y) = 1$. In the previous section we have found all solutions of our Thue equation under assumption that $U$ and $Z$ are relatively prime. Now we deal with the case when $\gcd(U, Z) = d > 1$.

Let $U = dU'$, $Z = dZ'$. Then $Z'^2 - (n^2 - 2)U'^2 = \frac{2m}{d^2}$, $\gcd(U', Z') = 1$ and $|\frac{2m}{d^2}| \leq n + \frac{3}{4}$. As in the proof of Proposition 1, we find that the only possibilities for $m' = \frac{2m}{d^2}$ are: $m' = 1$ and $m' = 2$ if $n \geq 4$, and $m' = 1$, $m' = 2$ and $m' = -3$ if $n = 3$. For $n = 3$, the additional possibility $m' = -3$, which implies $m = -6$, will be treated in the next section.

Consider first the case $m' = 2$. By Proposition 1 we find that the solutions $Z'$ of the equation $Z'^2 - (n^2 - 2)U'^2 = 2$ are given by $Z' = z_k$, where $z_0 = n$, $z_1 = 2n^3 - 3n$, $z_k = 2(n^2 - 1)z_{k-1} - z_{k-2}$. It follows easily by induction that $Z' = nz$, where $z$ is an odd integer. Now from $Z = |n(x^2 - y^2)| = dnz$ and $U = x^2 + y^2 = dU'$ it follows that $d$ divides $x^2 + y^2$ and $x^2 - y^2$. But $\gcd(x, y) = 1$ and thus $d = 2$. However, this implies that $2z = x^2 - y^2$, which is a contradiction since $2z \equiv 2 \pmod 4$ and so $2z$ cannot be represented as a difference of two squares.

Assume now that $m' = 1$. We have the Pell equation $Z'^2 - (n^2 - 2)U'^2 = 1$ with the solutions $Z'$ given by $Z' = w_k$, where $w_0 = 1$, $w_1 = n^2 - 1$, $w_k = 2(n^2 - 1)w_{k-1} - w_{k-2}$. Hence, $Z' \equiv \pm 1 \pmod n$ and $\gcd(Z', n) = 1$. But now from $Z = |n(x^2 - y^2)| = dZ'$ we obtain that $n|d$. Thus $d \geq n$ and for $n \geq 6$ we have $2m = d^2 \geq n^2 > 2n + 3$, and we obtain a contradiction again. For $n = 3$ and $n = 4$ there is an additional possibility that $m = 8$, which will be treated in the next section.

## 6 Cases $n \leq 6$

It remains to consider four particular Thue equations listed in Proposition 1 for $n = 3, 4, 5, 6$ and also the equations for $(n, m) = (3, -6)$, $(3, 8)$, $(4, 8)$ from the previous section. We can solve the corresponding systems of Pellian equations by standard methods (linear forms in three logarithms and Baker-Davenport reduction, see e.g. [2]).

But we can also use the Thue equation solver in PARI/GP [13] to solve directly these seven Thue equations. We easily find that the Thue equations have no integer solutions. Note that the corresponding system for $(n, m) =$

$(4, -7)$:

$$\begin{aligned} U^2 - 2V^2 &= -7, \\ Z^2 - 14U^2 &= -14, \end{aligned}$$

has an integer solution $(U, V, Z) = (1, 2, 0)$, but this solution does not lead to a solution of the corresponding Thue equation.

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