An absolute bound for the size of Diophantine m-tuples

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Abstract

A set of m positive integers is called a Diophantine m-tuple if the product of its any two distinct elements increased by 1 is a perfect square. We prove that if $\{a, b, c\}$ is a Diophantine triple such that $b > 4a$ and $c > \max\{b^{13}, 10^{20}\}$ or $c > \max\{b^5, 10^{1029}\}$, then there is unique positive integer d such that $d > c$ and $\{a, b, c, d\}$ is a Diophantine quadruple. Furthermore, we prove that there does not exist a Diophantine 9-tuple and that there are only finitely many Diophantine 8-tuples.

1 Introduction

A set of m positive integers $\{a_1, a_2, \ldots, a_m\}$ is called *a Diophantine m-tuple* if $a_i \cdot a_j + 1$ is a perfect square for all $1 \leq i < j \leq m$. The problem of construction of such sets was studied already by Diophantus who found a set of four positive rationals with the above property. However, the first Diophantine quadruple, $\{1, 3, 8, 120\}$, was found by Fermat (see [6]).

A famous conjecture is that there does not exist a Diophantine quintuple. There is even a stronger version of this conjecture, namely that if we fix a Diophantine triple $\{a, b, c\}$, then there is unique positive integer d such that $d > \max\{a, b, c\}$ and $\{a, b, c, d\}$ is a Diophantine quadruple. More precisely, let $\{a, b, c\}$ be a Diophantine triple. It follows that there exist positive integers r, s, t such that

$$ab + 1 = r^2, \quad ac + 1 = s^2, \quad bc + 1 = t^2. \tag{1}$$

Then it is easy to verify (see [1]) that for the positive integer d defined by

$$d = a + b + c + 2abc + 2rst$$

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we have:

$$ad + 1 = (at + rs)^2, \quad bd + 1 = (bs + rt)^2, \quad cd + 1 = (cr + st)^2.$$

Conjecture 1 *If $\{a, b, c, d\}$ is a Diophantine quadruple and $d > \max\{a, b, c\}$, then $d = a + b + c + 2abc + 2rst$.*

The first important result concerning this conjecture was proved in 1969 by Baker and Davenport [3] when they proved, answering a question of van Lint [16], Conjecture 1 for the Diophantine triple $\{1, 3, 8\}$. Later, Conjecture 1 was proved for certain other triples [21, 15] and for some parametric families of triples [7, 8, 10].

Since the number of integer points on an elliptic curve

$$y^2 = (ax + 1)(bx + 1)(cx + 1) \tag{2}$$

is finite, it follows that there does not exist an infinite set of positive integers with the property of Diophantus. However, bounds for the size [2, 12] and for the number [19] of solutions of (2) depend on parameters a, b, c and accordingly they do not immediately yield an absolute bound for the size of such set.

In the present paper we will prove Conjecture 1 for a large class of Diophantine triples, namely for triples satisfying some gap conditions like $b > 4a$, $c > \max\{b^{13}, 10^{20}\}$. This result will allow us to give an absolute bound for the size of Diophantine m-tuples.

Theorem 1 *There does not exist a Diophantine 9-tuple.*

Theorem 2 *There are only finitely many Diophantine 8-tuples.*

In the first part of the paper we follow the strategy from our papers [7, 8] and specially from the joint paper with A. Pethő [10]. We first transform the problem into solving systems of simultaneous Pellian equations, and this reduces to finding intersection of binary recursive sequences. We find necessary solvability conditions on initial terms of these sequences and using some congruence relations we obtain some lower bounds for the solutions. These lower bounds are compared with upper bounds which follow from a theorem of Bennett [5] on simultaneous approximations of algebraic numbers and a theorem of Baker and Wüstholz [4] on linear forms in logarithms of algebraic numbers. These comparisons actually complete the proof of Conjecture 1 for triples satisfying some technical gap conditions. In the final step, we deduce Theorems 1 and 2 combining previous results with some gap principles.

2 Systems of Pellian equations

Let $\{a, b, c\}$, where $0 < a < b < c$, be a Diophantine triple and let the positive integers r, s, t be defined by

$$ab + 1 = r^2, \quad ac + 1 = s^2, \quad bc + 1 = t^2.$$

In this paper, the symbols r, s, t will always have this meaning. Assume that $d > c$ is a positive integer such that $\{a, b, c, d\}$ is a Diophantine quadruple. We have

$$ad + 1 = x^2, \quad bd + 1 = y^2, \quad cd + 1 = z^2, \tag{3}$$

with positive integers x, y, z. Eliminating d from (3) we obtain the following system of Pellian equations

$$az^2 - cx^2 = a - c\,, \tag{4}$$
$$bz^2 - cy^2 = b - c\,. \tag{5}$$

We will now describe the sets of solutions of equations (4) and (5). We will follow arguments of Nagell [18, Theorem 108a]. Observe that $s+\sqrt{ac}$ and $t+\sqrt{bc}$ are non-trivial units of norm 1 in the number rings $\mathbf{Z}[\sqrt{ac}]$ and $\mathbf{Z}[\sqrt{bc}]$, respectively.

Lemma 1 *There exist positive integers i_0, j_0 and integers $z_0^{(i)}$, $x_0^{(i)}$, $z_1^{(j)}$, $y_1^{(j)}$, $i = 1, \ldots, i_0$, $j = 1, \ldots, j_0$, with the following properties:*

(i) $(z_0^{(i)}, x_0^{(i)})$ and $(z_1^{(j)}, y_1^{(j)})$ are solutions of (4) and (5), respectively.

(ii) $z_0^{(i)}$, $x_0^{(i)}$, $z_1^{(j)}$, $y_1^{(j)}$ satisfy the following inequalities

$$1 \le x_0^{(i)} \le \sqrt{\frac{a(c-a)}{2(s-1)}} < \sqrt{\frac{s+1}{2}} < \sqrt[4]{ac}\,, \tag{6}$$

$$1 \le |z_0^{(i)}| \le \sqrt{\frac{(s-1)(c-a)}{2a}} < \sqrt{\frac{c\sqrt{c}}{2\sqrt{a}}} < \frac{c}{2}\,, \tag{7}$$

$$1 \le y_1^{(j)} \le \sqrt{\frac{b(c-b)}{2(t-1)}} < \sqrt{\frac{t+1}{2}} < \sqrt[4]{bc}\,, \tag{8}$$

$$1 \le |z_1^{(j)}| \le \sqrt{\frac{(t-1)(c-b)}{2b}} < \sqrt{\frac{c\sqrt{c}}{2\sqrt{b}}} < \frac{c}{3}\,. \tag{9}$$

(iii) If (z, x) and (z, y) are positive integer solutions of (4) and (5) respectively, then there exist $i \in \{1, \ldots, i_0\}$, $j \in \{1, \ldots, j_0\}$ and integers $m, n \ge 0$ such that

$$z\sqrt{a} + x\sqrt{c} = (z_0^{(i)}\sqrt{a} + x_0^{(i)}\sqrt{c})(s + \sqrt{ac})^m, \tag{10}$$
$$z\sqrt{b} + y\sqrt{c} = (z_1^{(j)}\sqrt{b} + y_1^{(j)}\sqrt{c})(t + \sqrt{bc})^n. \tag{11}$$

Proof. It is clear that it suffices to prove the statement of the lemma for equation (4). Let (z,x) be a solution of (4) in positive integers. Consider all pairs (z^*,x^*) of integers of the form

$$z^*\sqrt{a}+x^*\sqrt{c}=(z\sqrt{a}+x\sqrt{c})(s+\sqrt{ac})^m, \quad m\in \mathbf{Z}\,.$$

It is clear that (z^*,x^*) satisfies (4) and that x^* is a positive integer. Among all pairs (z^*,x^*), we choose a pair with the property that x^* is minimal, and we denote that pair by (z_0,x_0). Define integers z' and x' by

$$z'\sqrt{a}+x'\sqrt{c}=(z_0\sqrt{a}+x_0\sqrt{c})(s-\varepsilon\sqrt{ac}),$$

where $\varepsilon=1$ if $z_0>0$, and $\varepsilon=-1$ if $z_0<0$. From the minimality of x_0 we conclude that $x'=sx_0-\varepsilon az_0\geq x_0$ and this leads to $a|z_0|\leq (s-1)x_0$. Squaring this inequality we obtain

$$acx_0^2+a(a-c)\leq (ac+2-2s)x_0^2$$

and finally

$$x_0^2\leq \frac{a(c-a)}{2(s-1)}\,.$$

Now we have

$$z_0^2=\frac{1}{a}(cx_0^2+a-c)\leq \frac{1}{a}\Big(\frac{ac(c-a)}{2(s-1)}+a-c\Big)=\frac{(s-1)(c-a)}{2a}\,. \tag{12}$$

Note that $a\geq 1$, $b\geq 3$ and $c\geq 8$. Hence, we have proved that there exists a solution (z_0,x_0) of (4) which satisfies (6) and (7) (and accordingly belongs to a finite set of solutions) and an integer $m\in\mathbf{Z}$ such that

$$z\sqrt{a}+x\sqrt{c}=(z_0\sqrt{a}+x_0\sqrt{c})(s+\sqrt{ac})^m.$$

It remains to show that $m\geq 0$. Suppose that $m<0$. Then $(s+\sqrt{ac})^m=\alpha-\beta\sqrt{ac}$, where α, β are positive integers satisfying $\alpha^2-ac\beta^2=1$. We have $z=\alpha z_0-\beta cx_0$ and from the condition $z>0$ we obtain $z_0^2>\beta^2c(c-a)\geq c(c-a)$ which clearly contradicts (12). ∎

From (10) we conclude that $z=v_m^{(i)}$ for some index i and integer $m\geq 0$, where

$$v_0^{(i)}=z_0^{(i)}, \quad v_1^{(i)}=sz_0^{(i)}+cx_0^{(i)}, \quad v_{m+2}^{(i)}=2sv_{m+1}^{(i)}-v_m^{(i)}\,, \tag{13}$$

and from (11) we conclude that $z=w_n^{(j)}$ for some index j and integer $n\geq 0$, where

$$w_0^{(j)}=z_1^{(i)}, \quad w_1^{(j)}=tz_1^{(j)}+cy_1^{(j)}, \quad w_{n+2}^{(j)}=2tw_{n+1}^{(j)}-w_n^{(j)}\,. \tag{14}$$

3 Congruence relations

The following lemma follows easily from (13) and (14) by induction.

Lemma 2

$$v_{2m}^{(i)} \equiv z_0^{(i)} \pmod{2c}, \quad v_{2m+1}^{(i)} \equiv sz_0^{(i)} + cx_0^{(i)} \pmod{2c},$$
$$w_{2n}^{(j)} \equiv z_1^{(j)} \pmod{2c}, \quad w_{2n+1}^{(j)} \equiv tz_1^{(j)} + cy_1^{(j)} \pmod{2c}.$$

Lemma 3

1) *If the equation $v_{2m}^{(i)} = w_{2n}^{(j)}$ has a solution, then $z_0^{(i)} = z_1^{(j)}$.*

2) *If the equation $v_{2m+1}^{(i)} = w_{2n}^{(j)}$ has a solution, then $z_0^{(i)} \cdot z_1^{(j)} < 0$ and $cx_0^{(i)} - s|z_0^{(i)}| = |z_1^{(j)}|$. In particular, if $b > 4a$ and $c > 100a$, then this equation has no solution.*

3) *If the equation $v_{2m}^{(i)} = w_{2n+1}^{(j)}$ has a solution, then $z_0^{(i)} \cdot z_1^{(j)} < 0$ and $cy_1^{(j)} - t|z_1^{(j)}| = |z_0^{(i)}|$.*

4) *If the equation $v_{2m+1}^{(i)} = w_{2n+1}^{(j)}$ has a solution, then $z_0^{(i)} \cdot z_1^{(j)} > 0$ and $cx_0^{(i)} - s|z_0^{(i)}| = cy_1^{(j)} - t|z_1^{(j)}|$.*

Proof.

1) From Lemmas 1 and 2 we have $|z_0^{(i)} - z_1^{(j)}| < c$ and $z_0^{(i)} \equiv z_1^{(j)} \pmod{2c}$, which implies $z_0^{(i)} = z_1^{(j)}$.

2) Observe that

$$cx_0^{(i)} - s|z_0^{(i)}| = \frac{c^2 - ac - (z_0^{(i)})^2}{cx_0^{(i)} + s|z_0^{(i)}|} < \frac{c^2 - s^2}{c + s} < c\,,$$

and since $c > 4a$ (see Lemma 14 below) we have also $c^2 - ac - (z_0^{(i)})^2 \geq \frac{1}{2}c^2 > 0$. Hence,

$$0 < cx_0^{(i)} - s|z_0^{(i)}| < c.$$

By Lemma 2 we have $sz_0^{(i)} + cx_0^{(i)} \equiv z_1^{(j)} \pmod{2c}$. Thus we conclude that if $z_0^{(i)} > 0$, then $z_1^{(j)} = sz_0^{(i)} - cx_0^{(i)}$, and if $z_0^{(i)} < 0$, then $z_1^{(j)} = sz_0^{(i)} + cx_0^{(i)}$.

Assume now that $b > 4a$ and $c > 100a$. Then

$$c^2 - ac - (z_0^{(i)})^2 \geq c^2 - \frac{c^2}{100} - \frac{c\sqrt{c}}{2\sqrt{a}} \geq 0.94c^2\,,$$

$$cx_0^{(i)} + s|z_0^{(i)}| < 2cx_0^{(i)} < 2c\sqrt{\frac{s+1}{2}} < 1.5c\sqrt[4]{ac}\,.$$

Thus, $cx_0^{(i)} - s|z_0^{(i)}| \geq 0.62\sqrt{\frac{c\sqrt{c}}{\sqrt{a}}}$. On the other hand, $|z_1^{(j)}| \leq \sqrt{\frac{c\sqrt{c}}{2\sqrt{b}}} \leq 0.5\sqrt{\frac{c\sqrt{c}}{\sqrt{a}}}$.

3) As in 2), we find that

$$0 < cy_1^{(j)} - t|z_1^{(j)}| < c\,,$$

which implies that if $z_1^{(j)} > 0$, then $z_0^{(i)} = tz_1^{(j)} - cy_1^{(j)}$, and if $z_1^{(j)} < 0$, then $z_0^{(i)} = tz_1^{(j)} + cy_1^{(j)}$.

4) We have already proved that

$$0 < cx_0^{(i)} - s|z_0^{(i)}| < c, \quad 0 < cy_1^{(j)} - t|z_1^{(j)}| < c.$$

From Lemma 2 we have

$$sz_0^{(i)} \pm cx_0^{(i)} \equiv tz_1^{(j)} \pm cy_1^{(j)} \pmod{2c}.$$

Hence, we have two possibilities: if $z_0^{(i)} > 0$, $z_1^{(j)} > 0$, then $sz_0^{(i)} - cx_0^{(i)} = tz_1^{(j)} - cy_1^{(j)}$, and if $z_0^{(i)} < 0$, $z_1^{(j)} < 0$, then $sz_0^{(i)} + cx_0^{(i)} = tz_1^{(j)} + cy_1^{(j)}$. ∎

In the following lemma we will consider the sequences $(v^{(i)} \bmod 4c^2)$ and $(w^{(j)} \bmod 4c^2)$. We will omit the superscripts (i) and (j) and we will continue to do so.

Lemma 4

1) $v_{2m} \equiv z_0 + 2c(az_0m^2 + sx_0m) \pmod{8c^2}$

2) $v_{2m+1} \equiv sz_0 + c[2asz_0m(m+1) + x_0(2m+1)] \pmod{4c^2}$

3) $w_{2n} \equiv z_1 + 2c(bz_1n^2 + ty_1n) \pmod{8c^2}$

4) $w_{2n+1} \equiv tz_1 + c[2btz_1n(n+1) + y_1(2n+1)] \pmod{4c^2}$

Proof. We give the proof of 1). The proofs of 2), 3) and 4) are completely analogous. We prove 1) by induction. We have $v_0 = z_0$ and $v_2 = z_0 + 2c(az_0 + sx_0)$. Assume that the assertion is valid for $m-1$ and m. The sequence (v_{2m}) satisfies the recurrence relation

$$v_{2m+2} = 2(2ac+1)v_{2m} - v_{2m-2}\,.$$

Hence,

$$\begin{aligned} v_{2m+2} &\equiv 4acz_0 + 2z_0 + 4c(az_0m^2 + sx_0m) - z_0 - 2c[az_0(m-1)^2 + sx_0(m-1)] \\ &\equiv z_0 + 2c[az_0(m+1)^2 + sx_0(m+1)] \pmod{8c^2}\,. \end{aligned}$$

∎

4 Linear forms in logarithms

Solving recurrences (13) and (14) we obtain

$$v_m = \frac{1}{2\sqrt{a}}[(z_0\sqrt{a}+x_0\sqrt{c})(s+\sqrt{ac})^m + (z_0\sqrt{a}-x_0\sqrt{c})(s-\sqrt{ac})^m], \tag{15}$$

$$w_n = \frac{1}{2\sqrt{b}}[(z_1\sqrt{b}+y_1\sqrt{c})(t+\sqrt{bc})^n + (z_1\sqrt{b}-y_1\sqrt{c})(t-\sqrt{bc})^n]. \tag{16}$$

Using the standard techniques (see [3, 11]) we will transform the equation $v_m = w_n$ into an inequality for linear form in three logarithms of algebraic numbers.

Lemma 5 *Assume that $c > 4b$. If $v_m = w_n$ and $m, n \neq 0$, then*

$$0 < m\log(s+\sqrt{ac}) - n\log(t+\sqrt{bc}) + \log\frac{\sqrt{b}(x_0\sqrt{c}+z_0\sqrt{a})}{\sqrt{a}(y_1\sqrt{c}+z_1\sqrt{b})} < \frac{8}{3}ac(s+\sqrt{ac})^{-2m}.$$

Proof. Put

$$P = \frac{1}{\sqrt{a}}(z_0\sqrt{a}+x_0\sqrt{c})(s+\sqrt{ac})^m, \quad Q = \frac{1}{\sqrt{b}}(z_1\sqrt{b}+y_1\sqrt{c})(t+\sqrt{bc})^n.$$

Then

$$P^{-1} = \frac{\sqrt{a}(x_0\sqrt{c}-z_0\sqrt{a})}{c-a}(s-\sqrt{ac})^m, \quad Q^{-1} = \frac{\sqrt{b}(y_1\sqrt{c}-z_1\sqrt{b})}{c-b}(t-\sqrt{bc})^n.$$

Therefore, the relation $v_m = w_n$ becomes $P - \frac{c-a}{a}P^{-1} = Q - \frac{c-b}{b}Q^{-1}$. As $m, n \geq 1$, we have

$$P \geq \frac{1}{\sqrt{a}}(x_0\sqrt{c}-|z_0|\sqrt{a})\cdot 2\sqrt{ac} = \frac{2\sqrt{c}(c-a)}{x_0\sqrt{c}+|z_0|\sqrt{a}} > \frac{\frac{3}{2}c\sqrt{c}}{c\sqrt{a}} = \frac{3}{2}\sqrt{\frac{c}{a}} > 1,$$

$$Q \geq \frac{2\sqrt{c}(c-b)}{y_1\sqrt{c}+|z_1|\sqrt{b}} > \frac{\frac{3}{2}c\sqrt{c}}{2\sqrt{c}\sqrt[4]{bc}} > 1.$$

Thus from

$$P - Q = \Big(\frac{c}{a}-1\Big)P^{-1} - \Big(\frac{c}{b}-1\Big)Q^{-1} > \Big(\frac{c}{a}-1\Big)(P^{-1}-Q^{-1}) = \Big(\frac{c}{a}-1\Big)(Q-P)P^{-1}Q^{-1}$$

it follows that $P > Q$. Furthermore, we have $\frac{P-Q}{P} < (\frac{c-a}{a})P^{-2}$. Since $P > \frac{3}{2}\sqrt{\frac{c}{a}}$, we have $\frac{c-a}{a}P^{-2} < \frac{4}{9} < \frac{1}{2}$, and finally we obtain (see [20, Lemma B.2])

$$\begin{aligned} 0 < \log\frac{P}{Q} &= -\log\Big(1 - \frac{P-Q}{P}\Big) \\ &< \frac{2(c-a)}{a}P^{-2} = \frac{2(c-a)}{a}\cdot\frac{a}{(z_0\sqrt{a}+x_0\sqrt{c})^2}(s+\sqrt{ac})^{-2m}. \end{aligned}$$

But

$$\frac{2(c-a)}{a}\cdot\frac{a}{(z_0\sqrt{a}+x_0\sqrt{c})^2}=\frac{2(z_0\sqrt{a}-x_0\sqrt{c})^2}{c-a}\le\frac{2(|z_0|\sqrt{a}+x_0\sqrt{c})^2}{c-a}\le\frac{2ac^2}{\frac{3}{4}c}=\frac{8}{3}ac\,.$$

■

Lemma 6 *Assume that $b > 4a$ and $c > 100b$. If $v_m = w_n$ and $n \ge 3$, then $m \ge n$.*

Proof. From Lemma 5 we have

$$\frac{m}{n}>\frac{\log(t+\sqrt{bc})}{\log(s+\sqrt{ac})}-\frac{\log\gamma}{n\log(s+\sqrt{ac})}\,,$$

where $\gamma=\frac{\sqrt{b}(x_0\sqrt{c}+z_0\sqrt{a})}{\sqrt{a}(y_1\sqrt{c}+z_1\sqrt{b})}$. Accordingly, it suffices to prove

$$\frac{\log\left(\frac{t+\sqrt{bc}}{s+\sqrt{ac}}\right)}{\log(s+\sqrt{ac})}-\frac{\log\gamma}{n\log(s+\sqrt{ac})}>-\frac{1}{n}\,,$$

and this is equivalent to

$$\left(\frac{t+\sqrt{bc}}{s+\sqrt{ac}}\right)^n>\frac{\gamma}{s+\sqrt{ac}}\,.$$

But

$$\frac{\gamma}{s+\sqrt{ac}}\le\frac{\sqrt{b}\cdot 1.45\sqrt{c}\sqrt[4]{ac}\cdot 1.45\sqrt{c}\sqrt[4]{bc}}{\sqrt{a}\cdot 0.99c\cdot 2\sqrt{ac}}\le 2.124\left(\frac{b}{a}\right)^{3/4},$$

and

$$\left(\frac{t+\sqrt{bc}}{s+\sqrt{ac}}\right)^3>\left(\frac{2\sqrt{bc}}{2.002\sqrt{ac}}\right)^3>0.997\left(\frac{b}{a}\right)^{3/2}.$$

Thus our condition becomes $(\frac{b}{a})^{3/4} > 2.131$ and this is clearly satisfied if $\frac{b}{a} > 4$. ■

Lemma 7 *Assume that $c > \max\{b^5, 10^{100}\}$ or $c > \max\{b^9, 10^{30}\}$ or $c > \max\{b^{13}, 10^{20}\}$. If $v_m = w_n$ and $n \ge 2$, then $m \le \frac{3}{2}n$.*

Proof. From Lemma 5 we obtain

$$\frac{m}{n}<\frac{\log(t+\sqrt{bc})}{\log(s+\sqrt{ac})}+\frac{\log\gamma}{n\log(s+\sqrt{ac})}+\frac{\frac{8}{3}ac(s+\sqrt{ac})^{-2m}}{n\log(s+\sqrt{ac})}\,. \tag{17}$$

We will estimate three summands on the right hand side of (17) separately and give details only for the case $c > \max\{b^5, 10^{100}\}$. We have

$$\frac{\log(t+\sqrt{bc})}{\log(s+\sqrt{ac})}=1+\frac{\log\left(\frac{t+\sqrt{bc}}{s+\sqrt{ac}}\right)}{\log(s+\sqrt{ac})}<1+\frac{\log\sqrt{\frac{b}{a}}}{\log(s+\sqrt{ac})}<1+\frac{1}{5}\,.$$

Assume that $n \geq 5$. From

$$\begin{aligned}\gamma &\leq \frac{\sqrt{b}(x_0\sqrt{c}+|z_0|\sqrt{a})}{\sqrt{a}(y_1\sqrt{c}-|z_1|\sqrt{b})} = \frac{\sqrt{b}(x_0\sqrt{c}+|z_0|\sqrt{a})(y_1\sqrt{c}+|z_1|\sqrt{b})}{\sqrt{a}(c-b)} \\ &\leq \frac{\sqrt{b}\cdot 2\sqrt{c}\sqrt[4]{ac}\cdot 2\sqrt{c}\sqrt[4]{bc}}{0.9c\sqrt{a}} < c^{0.657}\end{aligned}$$

we obtain

$$\frac{\log\gamma}{n\log(s+\sqrt{ac})} < \frac{0.657}{0.5n} < 0.263\,.$$

Finally,

$$\frac{\frac{8}{3}ac(s+\sqrt{ac})^{-2m}}{n\log(s+\sqrt{ac})} < \frac{\frac{8}{3}ac}{5\cdot 4ac\log(2\sqrt{ac})} < 0.002\,.$$

If we combine these three estimates, we obtain the desired result. In the remaining cases ($n \leq 4$), the above estimates give that if $n=1$, $n=2$, $n=3$, $n=4$ then $m \leq 2$, $m \leq 3$, $m \leq 4$, $m \leq 6$ respectively. ■

5 Lower bounds for solutions

In this section we will apply congruence relations from Section 3 to obtain some lower bounds for the indices m and n satisfying the equation $v_m = w_n$ with $n \neq 0, 1$. This method was introduced in our joint paper with A. Pethő [10], and it was also used in [9]. As Lemma 3 suggests, we will consider the equations $v_{2m} = w_{2n}$, $v_{2m} = w_{2n+1}$ and $v_{2m+1} = w_{2n+1}$ separately.

Lemma 8 *Assume that $b > 4a$.*

1) *If $v_{2m} = w_{2n}$, $n \neq 0$ and $c > \max\{b^{13}, 10^{20}\}$, then $m \geq n > c^{0.086}$.*
2) *If $v_{2m} = w_{2n}$, $n \neq 0$ and $c > \max\{b^{9}, 10^{30}\}$, then $m \geq n > c^{0.077}$.*
3) *If $v_{2m} = w_{2n}$, $n \neq 0$ and $c > \max\{b^{5}, 10^{100}\}$, then $m \geq n > c^{0.048}$.*

Proof. Lemmas 3 and 4 imply

$$az_0m^2 + sx_0m \equiv bz_0n^2 + ty_1n \pmod{4c}. \tag{18}$$

Let $c > \max\{b^{13}, 10^{20}\}$ and assume that $n \leq c^{0.086}$. Then by Lemma 7 we have $m < c^{0.095}$. We will estimate all four summands in (18):

$$\begin{aligned}|az_0m^2| &\leq a\sqrt{\frac{c\sqrt{c}}{2\sqrt{a}}}m^2 \leq c^{\frac{1}{13}\cdot\frac{3}{4}+\frac{3}{4}+2\cdot 0.095} < c\,, \\ sx_0m &\leq s\sqrt{\frac{s+1}{2}}m \leq \sqrt[4]{a^3c^3}m < c\,,\end{aligned}$$

$$\begin{aligned}
|bz_0n^2| &= |bz_1n^2| \le b\sqrt{\frac{c\sqrt{c}}{2\sqrt{b}}}n^2 \le c^{\frac{1}{13}\cdot\frac{3}{4}+\frac{3}{4}+2\cdot 0.086} < c\,,\\
ty_1n &\le \sqrt[4]{b^3c^3}n < c\,.
\end{aligned}$$

These estimates imply that in (18) we can replace $\equiv$ by $=$:

$$az_0m^2 + sx_0m = bz_0n^2 + ty_1n\,. \tag{19}$$

The same arguments can be used if $c > \max\{b^9, 10^{30}\}$, $n \le c^{0.077}$ and if $c > \max\{b^5, 10^{100}\}$, $n \le c^{0.048}$.

Let us consider the case when $|z_0| = 1$. Then $x_0 = y_1 = 1$ and (19) becomes

$$\pm am^2 + sm = \pm bn^2 + tn.$$

But $\max\{am, bn\} < c^{0.2} < 0.001\sqrt{c}$. Hence, $|\pm am^2 + sm| \le 1.001sm$ and $|\pm bn^2 + tn| \ge 0.999tn$. Therefore,

$$\frac{m}{n} \ge \frac{0.999}{1.001}\cdot\frac{t}{s} \ge \frac{0.999}{1.001^2}\cdot\sqrt{\frac{b}{a}} > 1.99\,,$$

which contradicts Lemma 7.

Thus we may assume that $|z_0| > 1$. Then $x_0 \ge 2$ and from $az_0^2 = cx_0^2 - c + a$ we obtain that $z_0^2 \ge \frac{3c}{a}$. This implies

$$0 \le \frac{sx_0}{a|z_0|} - 1 = \frac{x_0^2 + ac - a^2}{a|z_0|(sx_0 + a|z_0|)} \le \frac{ac}{2a^2z_0^2} \le \frac{1}{6}$$

and

$$0 \le \frac{ty_1}{b|z_1|} - 1 \le \frac{bc}{2b^2z_1^2} \le \frac{a}{6b} \le \frac{1}{24}\,.$$

If $z_0 > 1$, then by (19) we have $az_0m(m + \frac{7}{6}) \ge bz_0n(n+1)$ which for $m \ge 2$ implies $\frac{m}{n} \ge 1.59$ and for $m = n = 1$ gives $b \le 1.1a$, which are both impossible.

If $z_0 < -1$, then by (19) we have $a|z_0|m(m-1) \ge b|z_0|n(n - \frac{25}{24})$ which for $n \ge 3$ implies $\frac{m}{n} \ge 1.61$, while for $m = n = 2$ gives $b < 1.1a$ and for $n = 2$, $m = 3$ gives $b < 3.14a$. The only remaining case is $m = n = 1$. However, in this case we have

$$|z_0|(b-a)(sx_0 + ty_1) = c(b-a) + y_1^2 - x_0^2\,. \tag{20}$$

Since $ax_0^2 = by_1^2 - b + a > \frac{3}{4}by_1^2$, we have $x_0^2 > 3y_1^2$ and the right hand side of (20) is less that $c(b-a)$. It follows that $|z_0|(sx_0+ty_1) < c$, which is impossible since $|z_0|(sx_0+ty_1) \ge \sqrt{\frac{3c}{a}}\cdot 2\sqrt{ac} > c$. ■

Lemma 9 *Assume that $b > 4a$.*

1) *If $v_{2m} = w_{2n+1}$, $n \neq 0$ and $c > \max\{b^{13}, 10^{20}\}$, then $m \geq n > c^{0.0765}$.*

2) *If $v_{2m} = w_{2n+1}$, $n \neq 0$ and $c > \max\{b^{9}, 10^{30}\}$, then $m \geq n > c^{0.062}$.*

3) *If $v_{2m} = w_{2n+1}$, $n \neq 0$ and $c > \max\{b^{5}, 10^{100}\}$, then $m \geq n > c^{0.023}$.*

Proof. From Lemmas 3 and 4 we have

$$asz_0m^2 + sx_0m \equiv bz_0n(n+1) + y_1n' \pmod{2c}, \tag{21}$$

where $n' = n$ if $z_0 > 0$ and $n' = n+1$ if $z_0 < 0$.

Assume that $c > \max\{b^{13}, 10^{20}\}$ and $n \leq c^{0.0765}$. Then $n+1 < c^{0.0772}$ and $m+1 < \frac{3}{2}(n+1) < c^{0.0861}$. As in the proof of Lemma 8, we have

$$|az_0m^2| < c, \qquad sx_0m < c,$$

and also

$$\begin{aligned} |bz_0n(n+1)| &\leq b\sqrt{\frac{c\sqrt{c}}{2\sqrt{a}}}(n+1)^2 \leq c^{\frac{1}{13}+\frac{3}{4}+2\cdot 0.0772} < c\,, \\ y_1n' &\leq \sqrt[4]{bc}(n+1) < c\,. \end{aligned}$$

Hence, we have an equality in (21):

$$asz_0m^2 + sx_0m = bz_0n(n+1) + y_1n'\,,$$

and the same is true for $c > \max\{b^9, 10^{30}\}$, $n \leq c^{0.062}$ and for $c > \max\{b^5, 10^{100}\}$, $n \leq c^{0.023}$.

Note that $|z_0| = 1$ is impossible since it implies

$$c^2y_1^2 = t^2z_1^2 + 2tz_1 + 1 = bcz_1^2 + z_1^2 + 2tz_1^2 + 1$$

and

$$(z_1 + t)^2 = c(cy_1^2 - bz_1^2 + b) = c^2.$$

But $z_1 = \pm c - t$ is impossible since $|\pm c - t| > \sqrt{\frac{c\sqrt{c}}{2\sqrt{b}}}$.

Therefore we have $|z_0| \geq \sqrt{\frac{3c}{a}}$ and

$$1 \leq \frac{sx_0}{a|z_0|} \leq \frac{7}{6}\,, \qquad 0 < \frac{y_1}{b|z_0|} \leq \frac{\sqrt[4]{bc}}{b\sqrt{\frac{3c}{a}}} \leq \frac{1}{\sqrt[4]{c}} < 0.001\,.$$

If $z_0 > 0$, from (21) we obtain $az_0m(m+\frac{7}{6}) \geq bz_0n(n+1)$. For $m = 2$, $n = 1$ we get $\frac{b}{a} \leq 3.17$, and for $m \geq 3$ and $n \geq 2$ from $m(m+\frac{7}{6}) \leq 1.39m^2$, $n(n+1) \geq 0.96(n+\frac{1}{2})^2$ we obtain $\frac{2m}{2n+1} > 1.66$, which contradicts Lemma 7.

If $z_0 < 0$, from (21) we obtain $az_0m^2 \geq bz_0(n+1)(n-0.001)$ which implies $\frac{2m}{2n+1} > 1.88$. ∎

Lemma 10 *Assume that $b > 4a$.*

1) *If $v_{2m+1} = w_{2n+1}$, $n \neq 0$ and $c > \max\{b^{13}, 10^{20}\}$, then $m \geq n > c^{0.0765}$.*

2) *If $v_{2m+1} = w_{2n+1}$, $n \neq 0$ and $c > \max\{b^{9}, 10^{30}\}$, then $m \geq n > c^{0.062}$.*

3) *If $v_{2m+1} = w_{2n+1}$, $n \neq 0$ and $c > \max\{b^{5}, 10^{100}\}$, then $m \geq n > c^{0.023}$.*

Proof. Since $sz_0 \pm cx_0 = tz_1 \pm cy_1$, Lemma 4 implies

$$asz_0m(m+1) + x_0m' \equiv btz_1n(n+1) + y_1n' \pmod{2c}, \tag{22}$$

where $m' = m$, $n' = n$ if $z_0 < 0$ and $m' = m+1$, $n' = n+1$ if $z_0 > 0$. Noticing that $tz_1 \equiv sz_0 \pmod{c}$ and multiplying (22) by s or t, respectively, we obtain

$$\begin{aligned} az_0m(m+1) + sx_0m' &\equiv bz_0n(n+1) + sy_1n' \pmod{2c}, & (23)\\ az_1m(m+1) + tx_0m' &\equiv bz_1n(n+1) + ty_1n' \pmod{2c}. & (24) \end{aligned}$$

Assume that $c > \max\{b^{13}, 10^{20}\}$ and $n \leq c^{0.0765}$. Then $m + 1 < \frac{3}{2}(n+1) < c^{0.0861}$. This implies that all summands in (23) and (24) are less than c. Thus we actually have equalities in (23) and (24):

$$\begin{aligned} az_0m(m+1) + sx_0m' &= bz_0n(n+1) + sy_1n' \pmod{2c}, & (25)\\ az_1m(m+1) + tx_0m' &= bz_1n(n+1) + ty_1n' \pmod{2c}. & (26) \end{aligned}$$

We come to the same conclusion if $c > \max\{b^9, 10^{30}\}$, $n \leq c^{0.062}$ and if $c > \max\{b^5, 10^{100}\}$, $n \leq c^{0.023}$.

Suppose now that $sz_1 \neq tz_0$. Then (25) and (26) imply

$$\begin{aligned} am(m+1) &= bn(n+1), & (27)\\ tm' &= sn' & (28) \end{aligned}$$

But (27) implies $\frac{m}{n} \geq \sqrt{\frac{b}{a}} > 2$. On the other hand, for $n \geq 2$, by Lemma 7 we have

$$\frac{m}{n} \leq \frac{2m+1}{2n+1} \cdot \frac{5}{4} \leq \frac{3}{2} \cdot \frac{5}{4} < 2\,,$$

and for $n = 1$ we have $m = 1$ and $\frac{m}{n} = 1 < 2$.

Let us now consider the case $sz_1 = tz_0$. It is clear that $|z_0| \neq 1$ and $|z_1| \neq 1$. Hence, $|z_0| \geq \sqrt{\frac{3c}{a}}$, $|z_1| \geq \sqrt{\frac{3c}{b}}$ and we have

$$1 \leq \frac{sx_0}{a|z_0|} \leq \frac{7}{6}\,, \qquad 1 \leq \frac{sy_1}{b|z_0|} = \frac{ty_1}{b|z_1|} \leq \frac{7}{6}\,.$$

If $z_0 > 0$, from (25) we obtain $az_0(m+1)(m+\frac{7}{6}) \geq bz_0(n+1)^2$ which implies $\frac{2m+1}{2n+1} \geq \frac{m+1}{n+1} > 1.58$.

If $z_0 < 0$, we obtain $az_0m^2 \geq bz_0n(n-\frac{1}{6})$ which for $m = n = 1$ gives $\frac{b}{a} \leq 1.2$ and for $n \geq 2$ implies $\frac{m}{n} \geq 1.91$. But we have already seen that $\frac{m}{n} \leq \frac{15}{8} = 1.875$. ■

6 Application of a theorem of Bennett

Now we will apply the following important and very useful result of Bennett [5].

Lemma 11 *If a_i, p_i, q and N are integers for $0 \le i \le 2$, with $a_0 < a_1 < a_2$, $a_j = 0$ for some $0 \le j \le 2$, q nonzero and $N > M^9$, where*

$$M = \max_{0 \le i \le 2} \{|a_i|\},$$

then we have

$$\max_{0 \le i \le 2} \left\{ \left| \sqrt{1 + \frac{a_i}{N}} - \frac{p_i}{q} \right| \right\} > (130N\gamma)^{-1} q^{-\lambda}$$

where

$$\lambda = 1 + \frac{\log(33N\gamma)}{\log\left(1.7N^2 \prod_{0 \le i < j \le 2} (a_i - a_j)^{-2}\right)}$$

and

$$\gamma = \begin{cases} \frac{(a_2-a_0)^2(a_2-a_1)^2}{2a_2-a_0-a_1} & \textit{if } a_2 - a_1 \ge a_1 - a_0, \\ \frac{(a_2-a_0)^2(a_1-a_0)^2}{a_1+a_2-2a_0} & \textit{if } a_2 - a_1 < a_1 - a_0. \end{cases}$$

This is a result about simultaneous approximations of square roots of two rationals which are very close to 1, and it improves a result from [17]. We will apply Lemma 11 to the numbers

$$\theta_1 = \frac{s}{a}\sqrt{\frac{a}{c}} \quad \text{and} \quad \theta_2 = \frac{t}{b}\sqrt{\frac{b}{c}}\,.$$

Namely,

$$\begin{aligned} \theta_1 &= \sqrt{\frac{(ac+1)a}{a^2c}} = \sqrt{1 + \frac{1}{ac}} = \sqrt{1 + \frac{b}{abc}}\,, \\ \theta_2 &= \sqrt{1 + \frac{1}{bc}} = \sqrt{1 + \frac{a}{abc}}\,, \end{aligned}$$

so that θ_1 and θ_2 are indeed square roots of rationals which are close to 1. On the other hand, we will see in the next lemma that every solution of our problem induce good approximations of numbers θ_1 and θ_2.

Lemma 12 *All positive integer solutions x, y, z of the simultaneous Pellian equations*

$$\begin{aligned} az^2 - cx^2 &= a - c \\ bz^2 - cy^2 &= b - c \end{aligned}$$

satisfy

$$\max\left(\left|\theta_1 - \frac{sbx}{abz}\right|, \left|\theta_2 - \frac{tay}{abz}\right|\right) < \frac{c}{2a} \cdot z^{-2}\,.$$

Proof.

$$\begin{aligned}
\Big|\frac{s}{a}\sqrt{\frac{a}{c}}-\frac{sbx}{abz}\Big| &= \frac{s}{az\sqrt{c}}|z\sqrt{a}-x\sqrt{c}| = \frac{s}{az\sqrt{c}}\cdot\frac{c-a}{z\sqrt{a}+x\sqrt{c}} < \frac{s(c-a)}{2a\sqrt{ac}z^2} < \frac{c}{2a}\cdot z^{-2} \\
\Big|\frac{t}{b}\sqrt{\frac{b}{c}}-\frac{tay}{abz}\Big| &= \frac{t}{bz\sqrt{c}}\cdot\frac{c-b}{z\sqrt{b}+y\sqrt{c}} < \frac{t(c-b)}{2b\sqrt{bc}z^2} < \frac{c}{2b}\cdot z^{-2}
\end{aligned}$$

■

Corollary 1 *Let $\{a,b,c,d\}$, $a<b<c<d$ be a Diophantine quadruple.*

1) *If $c>\max\{b^{13},10^{20}\}$, then $d<c^{26}$.*

2) *If $c>\max\{b^{9},10^{30}\}$, then $d<c^{62}$.*

Proof. We give details for the proof of 1). The proof of 2) is completely analogous.

Let $ad+1=x^2$, $bd+1=y^2$, $cd+1=z^2$. We will apply Lemma 11 with $a_0=0$, $a_1=a$, $a_2=b$, $N=abc$, $M=b$, $q=abz$, $p_1=sbx$, $p_2=tay$. Since $abc>b^9$, the condition $N>M^9$ is satisfied. For the quantity γ from Lemma 11 we obtain $\gamma=\frac{b^2(b-a)^2}{2b-a}$ if $b\ge 2a$, and $\gamma=\frac{b^2a^2}{a+b}$ if $a<b<2a$. Considering γ as a function of a, we get in both cases that

$$\frac{b^3}{6}\le\gamma<\frac{b^3}{2}\,.$$

Furthermore, we have

$$\lambda=1+\frac{\log(33abc\gamma)}{\log(1.7c^2(b-a)^{-2})}=2-\lambda_1\,,$$

where

$$\lambda_1=\frac{\log\big(\frac{1.7c}{33ab(b-a)^2\gamma}\big)}{\log(1.7c^2(b-a)^{-2})}\,.$$

Lemma 11 and Lemma 12 imply

$$\frac{c}{2az^2}>(130abc\gamma)^{-1}(abz)^{\lambda_1-2}.$$

This implies

$$z^{\lambda_1}<65a^2b^3c^2\gamma$$

and

$$\log z<\frac{\log\big(65a^2b^3c^2\gamma\big)\log\big(1.7c^2(b-a)^{-2}\big)}{\log\big(\frac{1.7c}{33ab(b-a)^2\gamma}\big)}\,. \tag{29}$$

We will estimate the right hand side of (29). We have

$$\begin{aligned}
65a^2b^3c^2\gamma &< 32.5a^2b^6c^2 < 32.5b^8c^2 < c^{2.691},\\
1.7c^2(b-a)^{-2} &< c^2,\\
\frac{1.7c}{33ab(b-a)^2\gamma} &> \frac{1.7c}{16.5ab^6} > 0.103cb^{-7} > c^{0.412}.
\end{aligned}$$

Putting these estimates together, we obtain

$$\log z < \frac{2\cdot 2.691\log^2 c}{0.412\log c} < 13.07\log c\,.$$

Hence,

$$z < c^{13.07} \tag{30}$$

and

$$d = \frac{z^2-1}{c} < c^{25.14}.$$

■

7 The proof of the special case of Conjecture 1

Theorem 3 *If $\{a,b,c,d\}$ is a Diophantine quadruple such that $b > 4a$, $c > \max\{b^{13}, 10^{20}\}$ or $c > \max\{b^9, 10^{30}\}$, and $d > c$, then $d = a+b+c+2abc+2rst$.*

Proof. Let $ad+1 = x^2$, $bd+1 = y^2$, $cd+1 = z^2$. Then there exist integers $m, n \geq 0$ such that

$$z = v_m = w_n\,, \tag{31}$$

where the sequences (v_m) and (w_n) are defined by (13) and (14).

Assume that $n > 1$. Let $c > \max\{b^{13}, 10^{20}\}$. Then Lemmas 8, 9 and 10 imply $n > 2\cdot c^{0.0765}$. On the other hand, since

$$y_1\sqrt{c} - |z_1|\sqrt{b} = \frac{c-b}{|z_1|\sqrt{b} + y_1\sqrt{c}} > \frac{0.45\sqrt{c}}{\sqrt[4]{bc}} > 3\sqrt{b}\,,$$

it follows from (16) that

$$z = w_n > (t+\sqrt{bc})^n > (bc)^{n/2} > c^{n/2}\,.$$

Thus from (30) we conclude that $n \leq 26$. If we compare this with $n > 2\cdot c^{0.0765}$, we obtain $c < 4\cdot 10^{14}$, which is a contradiction. Similarly, for $c > \max\{b^9, 10^{30}\}$ we obtain $n \leq 62$ and $n > 2\cdot c^{0.062}$ which lead to $c < 2\cdot 10^{24}$, a contradiction.

Hence, we have $n \leq 1$. Let us consider three possible cases separately.

1) m and n are even. It follows that $m = n = 0$. Then $z = z_0 < c$ and $d = \frac{z^2-1}{c} < c$.

2) m is even and n is odd. It follows that $n = 1$ and $m = 0$ or 2. If $m = 0$, then $z = z_0 < c$ and $d < c$. If $m = 2$, then $z_0 < 0$ and Lemma 3 implies

$$z_0 = tz_1 - cy_1 . \tag{32}$$

Since $v_2 = z_0 + 2c(az_0 + sx_0)$, $v_2 = w_1$ implies

$$az_0 + sx_0 = y_1 . \tag{33}$$

Combining the conditions (32) and (33) with $az_0^2 - cx_0^2 = a - c$ and $bz_1^2 - cy_1^2 = b - c$, we obtain that

$$z_1 = s, \quad y_1 = r, \quad z_0 = st - cr, \quad x_0 = rs - at.$$

However,

$$|z_0| = \frac{c^2 - ac - bc - 1}{cr + st} > \frac{0.9c^2}{2c\sqrt{ab}} > \sqrt{\frac{c\sqrt{c}}{2\sqrt{a}}} ,$$

a contradiction.

3) m and n are odd. It follows that $m = n = 1$.

If $z_0 < 0$, then $z = v_1 = sz_0 + cx_0 < c$ and $d < c$.

If $z_0 > 0$, then

$$sz_0 - cx_0 = tz_1 - cy_1 , \tag{34}$$

by Lemma 3. On the other hand, $z = v_1 = w_1$ implies

$$sz_0 + cx_0 = tz_1 + cy_1 . \tag{35}$$

Clearly, (34) and (35) imply $x_0 = y_1$ and $sz_0 = tz_1$. If we put this in $az_0^2 - cx_0^2 = a - c$ and $bz_1^2 - cy_1^2 = b - c$, we obtain

$$(b - a)s^2 = (bz_1^2 - az_0^2)s^2 = z_1^2(abc + b - abc - a) = z_1^2(b - a).$$

It implies that $z_1 = s$, $z_0 = t$, $x_0 = y_1 = r$. Hence, $z = v_1 = st + cr$ and

$$d = \frac{z^2 - 1}{c} = \frac{abc^2 + ac + bc + 1 + 2rst + abc^2 + c^2 - 1}{c} = a + b + c + 2abc + 2rst.$$

∎

8 Some gap principles

As we have seen, the number $d = a+b+c+2abc+2rst$ has the property that all numbers $ad+1$, $bd+1$, $cd+1$ are perfect squares. Of course, the same is true also for the number $e = a + b + c + 2abc - 2rst$. The trivial observation that if $e \neq 0$, then $e \geq 1$ leads to the very useful gap principle.

Lemma 13 *If $\{a,b,c\}$ is a Diophantine triple and $a<b<c$, then $c=a+b+2r$ or $c\geq 4ab$.*

Proof. See [14], Lemma 4. ■

Lemma 14

1) *If $\{a,b,c\}$ is a Diophantine triple and $a<b<c$, then $c>4a$.*

2) *If $\{a,b,c,d\}$ is a Diophantine quadruple and $a<b<c<d$, then $d\geq 4bc$.*

Proof.

1) From Lemma 13 we have $c\geq a+b+2r>4a$.

2) If we apply Lemma 13 to the triples $\{a,c,d\}$ and $\{b,c,d\}$ we obtain that $d=a+c+2s$ or $d\geq 4ac$, $d=b+c+2t$ or $d\geq 4bc$ respectively. However,

$$a+c+2s<b+c+2t<4c\leq 4ac$$

implies that $d\geq 4bc$. ■

Lemma 15

1) *If $\{a,b,c\}$ is a Diophantine triple and $a<b<4a$, then $c=c_k$ or $c=\bar{c}_k$ for some $k\geq 0$, where the sequences (c_k) and $\bar{c}_k$ are given by*

$$c_0=0,\quad c_1=a+b+2r,\quad c_{k+2}=(4ab+2)c_{k+1}-c_k+2(a+b), \tag{36}$$

$$\bar{c}_0=0,\quad \bar{c}_1=a+b-2r,\quad \bar{c}_{k+2}=(4ab+2)\bar{c}_{k+1}-\bar{c}_k+2(a+b). \tag{37}$$

If $b=a+2$, then $\bar{c}_{k+1}=c_k$ for $k\geq 0$.

2) *If c_ic_j+1 is a perfect square and $0<i<j$, then $j=i+1$ or $j\geq 3i+2$. If $\bar{c}_i\bar{c}_j+1$ is a perfect square and $1<i<j$, then $j=i+1$ or $j\geq 3i$.*

Proof.

1) See [14], Theorem 8.

2) See [13], Theorem 3. ■

Note that solving the recurrences (36) and (37) we obtain

$$c_k=\frac{1}{4ab}\Big[(\sqrt{a}+\sqrt{b})^2(r+\sqrt{ab})^{2k}+(\sqrt{a}-\sqrt{b})^2(r-\sqrt{ab})^{2k}-a-b\Big], \tag{38}$$

$$\bar{c}_k=\frac{1}{4ab}\Big[(\sqrt{a}-\sqrt{b})^2(r+\sqrt{ab})^{2k}+(\sqrt{a}+\sqrt{b})^2(r-\sqrt{ab})^{2k}-a-b\Big]. \tag{39}$$

9 Proof of Theorem 1

Assume that $\{a_1, a_2, \dots, a_9\}$ is a Diophantine 9-tuple and $a_1 < a_2 < \cdots < a_9$. We will consider two cases depending on whether $a_2 < 4a_1$ or $a_2 > 4a_1$.

Let us first consider the case $a_2 > 4a_1$. Then by Lemma 14 we have the following estimates

$$a_3 > a_2, \quad a_4 \geq 4a_3a_2 > 4a_2^2, \quad a_5 \geq 4a_4a_3 > 4^2a_2^3,$$
$$a_6 \geq 4a_5a_4 > 4^4a_2^5, \quad a_7 \geq 4a_6a_5 > 4^7a_2^8, \quad a_8 \geq 4a_7a_6 > 4^{12}a_2^{13}.$$

We will apply Theorem 3 to the quadruple $\{a_1, a_2, a_8, a_9\}$. It is clear that the conditions $a_2 > 4a_1$ and $a_8 > a_2^{13}$ are satisfied. We must check the condition $a_8 > 10^{20}$. But if $a_2 \geq 10$, then $a_8 > 4^{12} \cdot 10^{13} > 10^{20}$, and the only Diophantine pair $\{a_1, a_2\}$ which satisfies $4a_1 < a_2 \leq 10$ is the pair $\{1, 8\}$. However, in this case $a_3 \geq 1 + 8 + 6 = 15$, and we have $a_8 > 4^{12}a_3^8a_2^5 > 10^{20}$.

Thus, the conditions of Theorem 3 are satisfied and we have the conclusion that

$$a_9 = a_1 + a_2 + a_8 + 2a_1a_2a_8 + 2\sqrt{(a_1a_2 + 1)(a_1a_8 + 1)(a_2a_8 + 1)}\,.$$

This implies that $a_9 \leq 2(a_1 + a_2 + a_8 + 2a_1a_2a_8) < 4a_8(a_1a_2 + 1)$. On the other hand, Lemma 14 implies that $a_9 \geq 4a_8a_7$, and we obtain a contradiction.

Let us consider now the case $a_2 < 4a_1$. In this case we will apply Lemma 15. Let (c_k) and $(\bar{c}_k)$ be the sequences defined in Lemma 15 with $a = a_1$ and $b = a_2$.

Assume that $a_3 = c_1$. Then $a_3 > 4a_1$ and $a_3 < 4a_2$. Hence, we have

$$a_8 \geq 4^{13}a_3^8a_2^5 \geq 4^{13}a_3^8a_3^5 4^{-5} = 4^8a_3^{13}.$$

If $a_3 \geq 15$, then $4^8a_3^{13} > 10^{20}$. However, the only Diophantine triples $\{a_1, a_2, a_3\}$ such that $a_1 < a_2 < a_3 \leq 14$ are $\{1, 3, 8\}$ and $\{2, 4, 12\}$. In [3] and [21] it is proved that these triples cannot be extended to quintuples (these are special cases of the result from [7]). Therefore, we may apply Theorem 3 to the triple $\{a_1, a_3, a_8\}$. As before, we obtain that

$$a_9 \leq 4a_8(a_1a_3 + 1)\,,$$

which is in contradiction with $a_9 \geq 4a_8a_7$ and $a_7 \geq 4^7a_3^5a_2^3$.

Hence, we may assume that $a_3 > c_1$. It means that $a_3 \geq 4a_1a_2$. Lemma 15 implies that one of the sets

$$\{a_3, a_4, \dots, a_9\} \cap \{c_k \,:\, k \geq 2\} \quad \text{and} \quad \{a_3, a_4, \dots, a_9\} \cap \{\bar{c}_k \,:\, k \geq 2\}$$

has at least four elements. Denote that set by C' and its elements by $c'_{k_1} < c'_{k_2} < \cdots < c'_{k_j}$, where $c'_{k_i} = c_{k_i}$ or $\bar{c}_{k_i}$. Lemma 15 implies that $k_2 = k_1 + 1$ or $k_2 \geq 3k_1$, and also $k_3 = k_2 + 1$ or $k_3 \geq 3k_2$. However, $k_3 = k_2 + 1$ is impossible since it implies $c'_{k_3} \leq (4a_1a_2 + 2)c'_{k_2}$, contradicting the estimate $c'_{k_3} \geq 4c'_{k_2}c'_{k_1}$. Hence, we have $k_3 \geq 3k_2$ and $k_4 \geq 3k_3 \geq 9k_2$.

If $j \geq 5$, then also $k_5 \geq 3k_4 \geq 27k_2$. Using formulas (38) and (39) it is easy to check that $c'_{3k} \geq (c'_k)^3$. It follows that $c'_{k_5} \geq (c'_{k_2})^{27}$. Since $c'_{k_1} \geq 4a_1a_2$, we have $c'_{k_2} \geq 16a_1^2a_2^2 \geq a_2^4$ and $c'_{k_5} \geq 10^{50}$. Therefore we may apply Theorem 3 to the triple $\{a_1, c'_{k_1}, c'_{k_5}\}$. As before, we conclude that there does not exist a positive integer $d > c'_{k_5}$ such that

$$\{a_1, a_2, c'_{k_1}, c'_{k_2}, c'_{k_3}, c'_{k_4}, c'_{k_5}, d\}$$

is a Diophantine 8-tuple. It implies that $j \leq 5$. In particular, this finishes the proof in the case $a_2 = a_1 + 2$, since in this case by Lemma 15 we have $j = 7$.

But if $a_2 \neq a_1 + 2$, then $a_2 \geq 8$ and $c'_{k_4} \geq (c'_{k_2})^9 \geq 8^{36} > 10^{30}$. Therefore we may apply Theorem 3 to the triple $\{a_1, c'_{k_1}, c'_{k_4}\}$. After doing that, we conclude that $j = 4$ and $a_9 = c'_{k_4}$.

Let $C'' = \{a_3, a_4, \ldots, a_9\} \setminus C' = \{c''_{l_1}, c''_{l_2}, c''_{l_3}\}$, where $c''_{l_1} < c''_{l_2} < c''_{l_3}$ and $c''_{l_i} = c_{l_i}$ or $\bar{c}_{l_i}$. As in the proof of $k_3 \geq 3k_2$, we see that we cannot have $k_2 = k_1 + 1$ and $l_2 = l_1 + 1$ simultaneously. It follows that $k_3 \geq 9k_1$ or $l_3 \geq 9l_1$. On the other hand, if both inequalities $k_3 \geq 9k_1$ and $l_3 \geq 9l_1$ are satisfied, then

$$a_8 = \max\{c'_{k_3}, c''_{l_3}\} \geq \max\{(c'_{k_1})^9, (c''_{l_1})^9\} \geq a_4^9 \geq \bar{c}_3^9 \geq (16a_1^2a_2^2)^9 \geq a_2^{36} > 10^{30}$$

(we cannot have $a_3 = \bar{c}_2$, $a_4 = c_2$, because $c_2 < 16a_1a_2^2$ and $4\bar{c}_2a_2 \geq 16a_1a_2^2$). Thus, applying Theorem 3 to the triple $\{a_1, a_4, a_8\}$ contradicts the existence of a_9. Hence, we have $a_3 = c'_{k_1}$, $a_4 = c'_{k_1+1}$ or $a_3 = c''_{l_1}$, $a_4 = c''_{l_1+1}$, which implies

$$a_8 \geq a_5^9 > 10^{30},$$

and an application of Theorem 3 leads to a contradiction as before. ■

10 Application of a theorem of Baker and Wüstholz

We will now apply the theory of linear forms in logarithms of algebraic numbers. Namely, we have already transformed the equation $v_m = w_n$ in Lemma 5 into the inequality for a linear form in three logarithms of algebraic numbers. Thus, we have everything ready for the application of the following famous result of Baker and Wüstholz [4].

Lemma 16 *For a linear form $\Lambda \neq 0$ in logarithms of l algebraic numbers $\alpha_1, \ldots, \alpha_l$ with rational integer coefficients $b_1, \ldots, b_l$ we have*

$$\log \Lambda \geq -18(l+1)!\, l^{l+1}(32d)^{l+2}h'(\alpha_1)\cdots h'(\alpha_l)\log(2ld)\log B\,,$$

where $B = \max(|b_1|, \ldots, |b_l|)$, and where d is the degree of the number field generated by $\alpha_1, \ldots, \alpha_l$.

Here

$$h'(\alpha) = \frac{1}{d} \max\left(h(\alpha), |\log\,\alpha|, 1\right),$$

and $h(\alpha)$ denotes the standard logarithmic Weil height of α.

We will apply Lemma 16 to the form

$$\Lambda = m\log(s+\sqrt{ac}) - n\log(t+\sqrt{bc}) + \log\frac{\sqrt{b}(x_0\sqrt{c}+z_0\sqrt{a})}{\sqrt{a}(y_1\sqrt{c}+z_1\sqrt{b})}.$$

We have $l=3$, $d=4$, $B=m$,

$$\alpha_1 = s+\sqrt{ac}, \qquad \alpha_2 = t+\sqrt{bc},$$

$$\alpha_3 = \frac{\sqrt{b}(x_0\sqrt{c}+z_0\sqrt{a})}{\sqrt{a}(y_1\sqrt{c}+z_1\sqrt{b})},$$

and under assumptions $c > 4b$ and $m \geq 2$ we obtain

$$h'(\alpha_1) = \frac{1}{2}\log\alpha_1 < \frac{1}{2}\log c, \qquad h'(\alpha_2) = \frac{1}{2}\log\alpha_2 < \frac{1}{2}\log c,$$

$$\begin{aligned} h'(\alpha_3) &< \frac{1}{4}\log\left[a^2(c-b)^2\cdot 2\sqrt[4]{\frac{b^2c}{a}}\cdot\frac{16}{3}\sqrt[4]{\frac{b^3c^2}{a}}\cdot\sqrt{\frac{b}{a}}\cdot\frac{8}{3}\sqrt[4]{\frac{b^3c}{a^2}}\right] \\ &< \frac{1}{4}\log\left(\frac{256}{9}a^{1/2}b^{2.5}c^3\right) < 1.5\log c, \end{aligned}$$

$$\log\frac{8}{3}ac(s+\sqrt{ac})^{-2m} < -\frac{1}{2}m\log c.$$

Hence,

$$\frac{1}{2}m\log c < 3.822\cdot 10^{15}\cdot\frac{1}{2}\log c\cdot\frac{1}{2}\log c\cdot 1.375\log c\cdot\log m$$

and

$$\frac{m}{\log m} < 2.867\cdot 10^{15}\log^2 c. \tag{40}$$

Theorem 4 *If $\{a,b,c,d\}$ is a Diophantine quadruple such that $b>4a$, $c>\max\{b^5, 10^{1029}\}$ and $d>c$, then $d=a+b+c+2abc+2rst$.*

Proof. We have already deduced in the proof of Theorem 3 that $m \leq 1$ and $d > c$ imply $d=a+b+c+2abc+2rst$. Thus we may assume that $m \geq 2$. But we proved in Lemmas 8, 9 and 10 that under the assumptions of the theorem it holds $m > 2\cdot c^{0.023}$. If we put this in (40), we obtain

$$\frac{m}{\log^3 m} < 5.42\cdot 10^{18},$$

which implies $m < 9.2\cdot 10^{23}$ and finally $c < 10^{1029}$. ∎

11 Proof of Theorem 2

Let $\{a_1, a_2, \dots, a_8\}$ be a Diophantine 8-tuple such that $a_1 < a_2 < \cdots < a_8$. We have $a_3 > 4a_1$, and also from Lemma 14

$$a_4 > a_3, \quad a_5 > 4a_3^2, \quad a_6 > 4^2 a_3^3, \quad a_7 > 4^3 a_3^5.$$

If $a_3 > 10^{343}$, then we may apply Theorem 4 to the triple $\{a_1, a_3, a_7\}$. This gives that

$$a_8 = a_1 + a_3 + a_7 + 2a_1a_3a_7 + 2\sqrt{(a_1a_3+1)(a_1a_7+1)(a_3a_7+1)} \leq 4a_7(a_1a_3+1),$$

which contradicts $a_8 \geq 4a_7a_6$.

Hence, we proved that there are only finitely many triples $\{a_1, a_2, a_3\}$ which can be extended to a Diophantine 8-tuple, namely it is possible only for triples with $\max\{a_1, a_2, a_3\} \leq 10^{343}$. Since there are only finitely many integer points on the elliptic curve

$$y^2 = (a_1x+1)(a_2x+1)(a_3x+1)\,,$$

the proof is finished. ■

Remark 1 From the proof of Theorem 2 it follows that if $\{a_1, a_2, \dots, a_8\}$ is a Diophantine 8-tuple and $a_1 < a_2 < \cdots < a_8$, then $a_4 < 10^{514}$. Since $a_4 > 4a_2$, we may apply (40) with $a = a_1$, $b = a_2$, $c = a_4$. We obtain $m < 2 \cdot 10^{23}$, and it implies that $a_8 < 10^{10^{26}}$.

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