PRODUCTS OF FACTORIALS WHICH ARE POWERS

A. BÉRCZES, A. DUJELLA, L. HAJDU, N. SARADHA, AND R. TIJDEMAN

Dedicated to Jan-Hendrik Evertse on the occasion of his 60th birthday.

Abstract. Extending earlier research of Erdős and Graham, we consider the problem of products of factorials yielding perfect powers. On the one hand, we describe how the representability of ℓ-th powers behaves when the number of factorials is smaller than, equal to or larger than ℓ, respectively. On the other hand, we investigate the problem that, for which fixed $n = b_1$ it is possible to find integers $b_2, \dots, b_k$ at most b_1 (obeying certain conditions) such that $b_1!b_2! \cdots b_k!$ is a perfect power. Here we distinguish the cases where the factorials may be repeated or are distinct.

1. Introduction

A famous result of Erdős and Selfridge [3] from 1975 states that the product of a block of at least two positive integers cannot be a perfect power. In particular, this implies that the product of two factorials $b_1!b_2!$ cannot be a square if $b_1 - b_2 > 1$. As a continuation, in 1976, Erdős and Graham [4] made a systematic study of the problem of products of factorials being a square. At the end of their paper they wrote that they would extend their studies to products of factorials being ℓ-th powers with $\ell > 2$. We did not find any paper of Erdős and Graham or anybody else on this topic.

The aim of this paper is to extend the investigations in this direction. First we describe how the representability of ℓ-th powers behaves when the number of factorials is smaller than, equal to or larger than ℓ, respectively. Then, we investigate the problem that, for which fixed

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$n = b_1$ it is possible to find integers $b_2, \dots, b_k$ at most b_1 (obeying some natural conditions) such that $b_1!b_2! \cdots b_k!$ is a perfect power. Observe that for $\ell = 2$, it is pointless to allow exponents of the factorials $b_i!$, that is, every factorial occurs only to the power $\ell - 1 = 1$. Hence there are two natural ways to extend the problem investigated in [4]: we may allow that $b_i!$ appears to some power $h_i < \ell$, or we may strictly assume that every factorial appears only once. Sections 2-4 contain our statements and examples. The proofs are given in Section 5.

2. Relation between representability of ℓ-th powers and the number of factorials

Consider the equation

$$b_1! \cdots b_k! = y^\ell \tag{1}$$

in positive integers $b_1, \dots, b_k$, y and ℓ with $\ell \geq 2$. Without loss of generality we shall always assume that ℓ is a prime, unless it is stated otherwise. By collecting together equal factorials which occur to an ℓ-th power, and omitting all the b_i's which equal 1, we may rewrite the equation in the form

$$(a_1!)^{h_1} \cdots (a_k!)^{h_k} = y^\ell \tag{2}$$

with $a_1 > \cdots > a_k \geq 2$ and $1 \leq h_i < \ell$ $(1 \leq i \leq k)$. Set $N = h_1 + \cdots + h_k$. For later use, observe that $a_1 \geq k+1$ and is not a prime, and that $k \geq 2$, since otherwise $a_1!$ would be an ℓ-th power contradicting Bertrand's postulate.

Our first theorem describes the connection between the solvability of (2), and the relation between ℓ and N. This statement is a generalization of results of Erdős and Graham p. 342, [4], concerning the case $\ell = 2$ and serves as a kind of starting point for the next section.

Theorem 2.1. *Equation* (2) *with* $N < \ell$ *has no solutions. When* $N = \ell$, *the solutions of* (2) *are precisely those satisfying*

$$a_1 = x^\ell, a_2 = x^\ell - 1, h_1 + h_2 = \ell$$

where $x > 1$ *is an integer. Finally, equation* (2) *has infinitely many solutions for any* $N > \ell$.

Equation (2) with $\ell = 2$ has been studied by Erdős and Graham [4] in great detail. (Note that in that case we have $h_1 = \cdots = h_k = 1$ and $N = k$.) They showed that if $k = 2$, then a_1 has to be a square. Obviously, if $a_1 > 1$ is a square, then $a_1!(a_1-1)!$ is a square, too. When $k = 3$ and $a_1 - a_2 = 1, 2$, infinitely many solutions are shown in [4]. When $a_1 - a_2 = 3$ then only finitely many solutions are known. The list of solutions found in [4] has been extended by Dujella, Najman, Saradha and Shorey [2], by solving

$$a_1!a_2!a_3! = y^2 \tag{3}$$

completely for $a_3 < 100$.

As we see, in the last result above, one of the factorials is fixed. This motivates us to give the following statement, which is a partial extension of Theorem 2.1.

Theorem 2.2. *Let t be a given positive integer and ℓ a given prime. Consider the equation*

$$t(a_1!)^{h_1} \cdots (a_k!)^{h_k} = y^\ell \tag{4}$$

in positive integers $a_1, \dots, a_k$ and $h_1, \dots, h_k$ with $a_1 > \dots > a_k \geq 2$ $1 \leq h_i < \ell$ $(i = 1, \dots, k)$. Put $N = h_1 + \dots + h_k$. If $N < \ell$, or $N = \ell$ and

$$a_1 - a_k > \begin{cases} 2, & \text{if } \ell = 2, \\ 1, & \text{otherwise,} \end{cases}$$

then equation (4) has only finitely many solutions. Further, all these solutions can be effectively determined.

Remark. By Theorem 2.1, the condition that $N \leq \ell$ is clearly necessary. Further, in view of the above mentioned results of Erdős and Graham [4] for $\ell = 2$ and $k = 3$, and Theorem 2.1 in case of general ℓ, the condition prescribed for $a_1 - a_k$ when $N = \ell$ is also necessary.

Finally, we mention that it is also necessary to fix ℓ in this generality. This is shown by the infinitude of the solutions

$$t(x^2!)^{h_1}((x^2-1)!)^{h_2} = y^\ell$$

with ℓ being an odd prime, $t = y = x((x^2-1)!)$ and $h_1 = h_2 = (\ell-1)/2$ (whence $N < \ell$).

3. Products of factorials with fixed starting term yielding ℓ-th powers

This is the case where we can follow the arguments and treatment of Erdős and Graham [4]. First we extend their result concerning the case $\ell = 2$, to the general situation.

Theorem 3.1. *If n is not prime and $\ell \geq 2$, then there exist positive integers $n = a_1 > a_2 > \dots > a_k \geq 2$ and $h_1, \dots, h_k$ with $k \leq 6$ for which $1 \leq h_i \leq (\ell+1)/2$ $(i = 1, \dots, k)$ and $h_1 + \dots + h_k \leq 3\ell$, such that $(a_1!)^{h_1}(a_2!)^{h_2} \cdots (a_k!)^{h_k}$ is an ℓ-th power.*

Remark. It is obvious that in the above theorem the condition that n is not a prime is necessary. Moreover, if $n-1$ is prime, then the bound for the h_i's cannot be improved upon. Indeed, the prime $n-1$ occurs at most to the power h_1+h_2, while it should occur at least to the power ℓ. So h_1 or h_2 has to be at least $\ell/2$.

We introduce some notation similar to that in [4]. We shall also use the notation from the previous section.

For fixed ℓ, put $F_0^{(\ell)}=\emptyset$, and for $i=1,2,\dots$ set

$$F_i^{(\ell)}=\{n\ :\ (2) \text{ has a solution with } a_1=n \text{ and } N=N(\ell,n)\le i\}$$

and

$$D_i^{(\ell)}=F_i^{(\ell)}\setminus F_{i-1}^{(\ell)}.$$

Thus if $n\in D_i^{(\ell)}$, then $N(\ell,n)=i$. These sets in case of $\ell=2$ have been intensively studied by Erdős and Graham [4]. They showed that $D_k^{(2)}=\emptyset$ for $k>6$. They investigated the sets $D_k^{(2)}$ for $1\le k\le 6$ in detail. For instance they proved that $D_3^{(2)}$ is sparse by showing $|D_3^{(2)}(X)|=o(X)$ for any $X>1$. This has been recently improved by Luca, Saradha and Shorey [5] to

$$|D_3^{(2)}(X)|=O\left(\frac{X}{\exp(c(\log X)^{1/4}(\log\log X)^{3/4})}\right)$$

with some absolute constant c. The proofs of the above mentioned results in [4] and [5] depend upon estimates from prime number theory. We shall consider the case $\ell\ge 3$.

Observe that by the classical result of Erdős and Selfridge [3] mentioned in the introduction, we have $F_1^{(\ell)}=D_1^{(\ell)}=\{1\}$ for any ℓ. Further, by Theorem 2.1 we also have

$$F_2^{(\ell)}=\dots=F_{\ell-1}^{(\ell)}=\{1\},\ \text{whence } D_2^{(\ell)}=\dots=D_{\ell-1}^{(\ell)}=\emptyset,$$
$$F_\ell^{(\ell)}=\{x^\ell\ :\ x \text{ is a positive integer}\}$$

and

$$D_\ell^{(\ell)}=\{x^\ell\ :\ x \text{ is a positive integer greater than } 1\}.$$

As we observed earlier, if n is a prime then $n\notin F_i^{(\ell)}$ for any i. Thus the following result is an immediate consequence of Theorem 3.1.

Corollary 3.1. *Let n be a composite integer. Then there exists an i with $\ell\le i\le 3\ell$ such that $n\in D_i^{(\ell)}$.*

In what follows, we shall be concerned with problems involving $D_i^{(\ell)}$ and $F_i^{(\ell)}$ for the "extremal" choices of i, that is for $i=\ell+1$ and 3ℓ.

The following theorem shows that the set $\cap_{\ell\ \text{prime}} D_{3\ell}^{(\ell)}$ is not empty.

Theorem 3.2. *We have* $17 \times 419 \in \cap_{\ell \text{ prime}} D_{3\ell}^{(\ell)}$.

We think that $n \in F_{\ell+1}^{(\ell)}$ may be valid only in special cases. We pose the following

Problem 1. Let n be a given integer such that $n \in F_{\ell+1}^{(\ell)}$ for some ℓ being large enough in terms of n. Is it true that $n \in F_{\ell+1}^{(\ell)}$ for all ℓ?

The following result gives an affirmative answer to the above problem in a special case.

Theorem 3.3. *Let n be a positive integer such that $n-1$ is a prime and $n \in F_{\ell_0+1}^{(\ell_0)}$ for some $\ell_0 > \frac{n(\log n)^2}{(\log 2)^2}$. Then $n \in F_{\ell+1}^{(\ell)}$ for all ℓ.*

Remarks. 1. The set $\cap_{\ell \text{ prime}} D_{\ell+1}^{(\ell)}$ is infinite. Indeed, for any $a \geq 2$, $n = a!$ belongs to this set: on the one hand, by Theorem 2.1, $n \notin D_{\ell}^{(\ell)}$ for any $\ell \geq 2$, and on the other hand, as $((a!)!)^{\ell-1}(a!-1)!a! = (a!(a!-1)!)^{\ell}$, $n \in F_{\ell+1}^{(\ell)}$ for every $\ell \geq 2$.
2. The above theorem does not hold if $F_{\ell_0+1}^{(\ell_0)}$ and $F_{\ell+1}^{(\ell)}$ are replaced by $D_{\ell_0+1}^{(\ell_0)}$ and $D_{\ell+1}^{(\ell)}$, respectively. Indeed, let $n-1 = 2^p - 1$ be a Mersenne-prime. Further, choose a such that $\ell_0 := ap+1$ is a prime. (Clearly, ℓ_0 can be arbitrarily large.) Putting $b = ap - a + 1$, we have

$$(n!)^a((n-1)!)^b 2! = 2^{ap+1}((2^p-1)!)^{a+b} = (2 \cdot (2^p-1)!)^{\ell_0},$$

so $n \in F_{\ell_0+1}^{(\ell_0)}$. Since n is not an ℓ_0-th power, Theorem 2.1 shows that $n \notin D_{\ell_0}^{(\ell_0)}$, thus $n \in D_{\ell_0+1}^{(\ell_0)}$. On the other hand, we have by Theorem 2.1 that $n \in D_p^{(p)}$ and therefore $n \notin D_{p+1}^{(p)}$.

4. Products of distinct factorials with fixed starting term yielding ℓ-th powers

For any $n \geq 1$ write Q_ℓ for the ℓ-th power free part of n, with $Q_\ell(1) = 1$, for any $\ell \geq 2$.

In this section we are interested in the following

Problem 2. Let $\ell \geq 2$. For what $n \geq 1$ do there exist positive integers $n = a_1 > \cdots > a_k \geq 2$ such that $a_1! \cdots a_k!$ is an ℓ-th power?

Since $1! = 1^\ell$ for any ℓ, we shall assume that $n \geq 2$. Observe that if any of the numbers $n, n-1, \ldots, n-\ell+2$ is a prime, then n obviously does not have the required property.

The next theorem shows that for $\ell = 3$, apart from the above mentioned exceptions, all integers n have the property described in Problem 2.

Theorem 4.1. *Let $n > 1$ such that neither n nor $n-1$ is a prime. Then there exist positive integers $n = a_1 > \cdots > a_k \geq 2$ such that $a_1! \cdots a_k!$ is a perfect cube.*

The following result gives an upper bound for the number of factorials needed to get a cube starting from $n!$, for infinitely many values of n.

Theorem 4.2. *There exists infinitely many integers n such that for some positive integers $n = a_1 > \cdots > a_k \geq 2$ we have that $a_1! \cdots a_k!$ is a perfect cube, and $k < 1.5 \ln n$.*

Remark. Let $\ell \geq 4$ not necessarily a prime. Suppose that p is such a prime that $p+2$ is also a prime but none of $2p-1, 2p-2, \dots, 2p-\ell+1$ is a prime. If we take $n = 2p$, then n does not have the property described in Problem 2. Indeed, in $M := a_1! \cdots a_k!$ with positive integers $n = a_1 > \cdots > a_k \geq 2$, we clearly have $0 < \mathrm{ord}_p(M) - \mathrm{ord}_{p+2}(M) < 4$, where $\mathrm{ord}_q(m)$ denotes the exponent of the prime q in the prime factorization of the positive integer m. This shows that $a_1! \cdots a_k!$ cannot be an ℓ-th power with $\ell \geq 4$. Hence it seems to be a safe conjecture that for any $\ell \geq 4$ there are infinitely many n, such that none of $n, n-1, \dots, n-\ell+1$ is a prime, but n still does not have the property described in Problem 2. As interesting examples, we mention that

$$16! \cdot 15! \cdot 14! \cdot 13! \cdot 9! \cdot 5! \cdot 4! = 62768369664000^4,$$

and

$$28! \cdot 26! \cdot 25! \cdot 24! \cdot 23! \cdot 15! \cdot 14! \cdot 13! \cdot 12! \cdot 11! \cdot 9! \cdot 8! \cdot 6! \cdot 2! =$$
$$= 9736133659357487861493147893760000000^5.$$

5. Proofs

Later on, we shall use the following standard notation. For any integer $n > 1$ let $P(n)$ denote the greatest prime factor of n and put $P(1) = 1$. Further, let $p^{(n)}$ denote the least prime satisfying $p^{(n)} \geq n$.

To prove Theorems 2.1 and 2.2 the following deep result of Erdős and Selfridge [3] will be needed.

Lemma 5.1. *Let s, ℓ, m be integers such that $s \geq 3$, $\ell \geq 2$ and $m+s \geq p^{(s)}$. Then there is a prime $p \geq s$ for which $\ell \nmid \mathrm{ord}_p((m+1)\cdots(m+s))$.*

Proof of Theorem 2.1. Suppose first that $N < \ell$. Then for any a_1 by Bertrand's postulate, there exists a prime p with $a_1/2 < p \leq a_1$ and it is easy to see that p occurs at most to the first power in each $a_i!$ $(1 \leq i \leq k)$. Hence the order of p on the left hand side of (2) is at most $N < \ell$ which contradicts (2).

Assume next that $N > \ell$. Let $h = N - \ell$, and take arbitrary positive integers $c_1 > \cdots > c_h > 2$. Put

$$n = c_1! \cdots c_h!.$$

Observe that $n - 1 > c_1$. Then

$$(n!)^{\ell-1}(n-1)!c_1! \cdots c_h!$$

is a product of $\ell + h = N$ factorials yielding an ℓ-th power. Since $c_1, \dots, c_h$ can be chosen arbitrarily, (2) has infinitely many solutions in this case.

Finally, take $N = \ell$. Then after cancelling $(a_k!)^\ell$, (2) can be rewritten as

$$a_1^{u_1}(a_1-1)^{u_2} \cdots (a_k+1)^{u_{a_1-a_k}} = z^\ell \tag{5}$$

with some positive integers z and u_i with $1 \le u_i < \ell$ ($i = 1, \dots, a_1 - a_k$). If $a_1 < p^{(a_1-a_k)}$ would hold, then by Bertrand's postulate we would have $a_1 < 2(a_1 - a_k)$, thus also $a_1/2 > a_k$. Then by Bertrand's postulate again, we could find a prime q such that $a_k + 1 \le a_1/2 < q \le a_1$. However, then q would divide the left hand side of the above equation to the power u_i for some i with $1 \le i \le a_1 - a_k$, which is a contradiction. Thus we may assume that $a_1 \ge p^{(a_1-a_k)}$. Now if $a_1 - a_k > 2$ then by Lemma 5.1 we can take a prime p with $p > a_1 - a_k$ such that $\ell \nmid \mathrm{ord}_p(a_1 - j)$ for some j with $0 \le j < a_1 - a_k$. Then $p \mid a_1 - j$, but $p \nmid a_1 - j'$ for $0 \le j' < a_1 - a_k$ with $j' \ne j$. Thus (5) implies that $\ell \mid u_{j+1}\mathrm{ord}_p(a_1 - j)$. However, as $u_{j+1} < \ell$ and ℓ is a prime, this is a contradiction. On the other hand, if $a_1 - a_k = 2$, then (5) reads as $a_1^{u_1}(a_1-1)^{u_2} = z^\ell$. Since $\gcd(a_1^{u_1}, (a_1-1)^{u_2}) = 1$, from this we get that both $a_1^{u_1}$ and $(a_1-1)^{u_2}$ are ℓ-th powers, whence both a_1 and $a_1 - 1$ are ℓ-th powers, which is impossible. So we are left with $a_1 - a_k = 1$. Then (2) reduces to

$$(a_1!)^{h_1}((a_1-1)!)^{\ell-h_1} = z^\ell.$$

Thus $a_1^{h_1}$ must be an ℓ-th power. Since $1 \le h_1 < \ell$, we get $a_1 = x^\ell$ for some integer $x > 1$. On the other hand, one can readily check that for any a_1, a_2, h_1, h_2 with

$$a_1 = x^\ell, a_2 = x^\ell - 1, h_1 + h_2 = \ell$$

(2) holds. Hence the theorem follows. □

Proof of Theorem 2.2. Assume first that $N < \ell$. As $h_i < \ell$ for all $i = 1, \dots, k$, we may clearly assume that $a_1 > 2t$. Let p be a prime with $a_1/2 < p < a_1$; note that $p \nmid t$. Then, since the exponent of p

on the left hand side of (4) is at most N, the statement follows in this case.

Suppose next that $N = \ell$. Let first $P(t) \geq a_1 - a_k$. Then, letting $x = a_k + 1$ and $s = a_1 - a_k$, (4) can be reduced to

$$tx^{u_0}(x+1)^{u_1} \cdots (x+s-1)^{u_{s-1}} = z^\ell,$$

where $1 \leq u_i < N$ for $i = 0, \dots, s-1$. As $P(t) \geq s = a_1 - a_k$, the above equation is a superelliptic or hyperelliptic equation, and by a result of Brindza [1], the theorem follows. So we may suppose that $P(t) < s = a_1 - a_k$. Clearly, we may further assume that $P(t) < a_1/2$. Suppose that $a_1 < p^{(a_1-a_k)}$. Then by Bertrand's postulate $a_1 < 2(a_1 - a_k)$. Then again using Bertrand's postulate, there is a prime q such that $a_k + 1 \leq a_1/2 < q \leq a_1$. This q divides the left hand side of the above equation to the power u_i for some $i = 1, \dots, a_1 - a_k$, which is a contradiction. Thus we may assume that $a_1 \geq p^{(a_1-a_k)}$. Now if $s \geq 3$, then since ℓ is a prime, the theorem immediately follows from Lemma 5.1. If $s \leq 2$, then by our assumptions $s = 2$ and ℓ is an odd prime. In this case the statement again follows from the result of Brindza [1]. $\square$

Proof of Theorem 3.1. Let $n = p^2$ with p prime. If $\ell = 2$, then

$$n!(n-1)! = (p(n-1)!)^2$$

is a square. If ℓ is odd, then

$$(n!)^{(\ell-1)/2}((n-1)!)^{(\ell+1)/2}p!((p-1)!)^{\ell-1} = (p!(n-1)!)^\ell$$

is an ℓ-th power. Otherwise we can write $n = ab$ with $a > b > 1$. Then, for $\ell = 2$

$$n!(n-1)!a!(a-1)!b!(b-1)! = ((n-1)!a!b!)^2$$

is a square. If ℓ is odd, then

$$(n!)^{\frac{\ell+1}{2}}((n-1)!)^{\frac{\ell-1}{2}}(a!)^{\frac{\ell-1}{2}}((a-1)!)^{\frac{\ell+1}{2}}(b!)^{\frac{\ell-1}{2}}((b-1)!)^{\frac{\ell+1}{2}} = ((n-1)!a!b!)^\ell$$

is an ℓ-th power. $\square$

Proof of Theorem 3.2. We find it more convenient to use notation (1) with $b_1 \geq \cdots \geq b_k$ in our arguments. To prove the statement, by Theorem 3.1 it suffices to show that $n \notin F_{3\ell-1}^{(\ell)}$. For this, suppose that we have an equality of the form (1) with $b_1 = n = 17 \times 419$.

We have $n - 1 = 2 \times 3 \times 1187$, and $n - 2 = 7121$ is a prime. This shows that $b_\ell \in \{n-1, n-2\}$. We split the proof into several cases.

I) Assume first that $b_\ell = n - 2$. Then the ℓ-th power free part of $b_1! \cdots b_\ell!$ is

$$n^u(n-1)^v = 2^v \times 3^v \times 17^u \times 419^u \times 1187^v$$

with some $1 \leq u \leq v \leq \ell - 1$. This shows that $k > \ell$, and also that $b_{\ell+1} \geq 1187$ should be valid. Observe that if $b_{\ell+1} = n-2$, then similarly as above, we must also have

$$b_{\ell+2} = \cdots = b_{2\ell} = n - 2.$$

However, then as $n - 2 = b_\ell = b_{\ell+1} = \cdots = b_{2\ell}$, the term $(n-2)!$ would occur at least $\ell + 1$ times, which is impossible. So we must have $b_{\ell+1} < n - 2$.

I.1) Suppose that $n - 2 > b_{\ell+1} \geq 1193$, which is the smallest prime greater than 1187. Observe that $b_{\ell+1}$ cannot be a prime, since otherwise we should have $b_{\ell+1} = \cdots = b_{2\ell}$ which is not possible. So let p and q be two consecutive primes with $p > b_{\ell+1} > q$. Clearly, we must have $b_{\ell+1} \geq \cdots \geq b_{2\ell} \geq q$. We check the possible pairs (p, q) one by one. Write S for the set of prime divisors of the integers strictly between p and q.

If 1187 is not contained in S, then we see that $k > 2\ell$ must be valid, and $b_{2\ell+1} \geq 1187$ should hold. With a simple computer program we checked that for all such candidates for $b_{2\ell+1}$, there is a prime r outside $S \cup \{2, 3, 17, 419, 1187\}$ such that $b_{2\ell+1} > r > b_{2\ell+1}/2$. This gives that r divides the ℓ-th power free part of $b_1! \cdots b_{2\ell+1}!$ precisely on the first power. Hence we get that $k \geq 3\ell$ must be valid (and $b_{2\ell+2} \geq \cdots \geq b_{3\ell} \geq r$). For example, if $p = 1201$ and $q = 1193$, then

$$S = \{2, 3, 5, 7, 11, 13, 19, 23, 109, 199, 239, 599\},$$

and taking e.g. $r = 997$ for any possible choice of $b_{2\ell+1}$, our claim follows.

If $1187 \in S$, but 1187 does not divide $b_{\ell+1}(b_{\ell+1}-1) \cdots (b_{2\ell-1}+1)$, then a similar argument applies. For example, if $p = 2377$ and $q = 2371$, then

$$S = \{2, 3, 5, 7, 11, 19, 113, 593, 1187\}.$$

However, if

$$2377 > b_{\ell+1} \geq \cdots \geq b_{2\ell} \geq 2374 = 2 \times 1187$$

then 1187 does not divide the ℓ-th power free part of $b_{\ell+1}! \cdots b_{2\ell}!$. Thus similarly as before, we see that $k > 2\ell$ and $2374 \geq b_{2\ell+1} \geq 1187$. For all such values of $b_{2\ell+1}$ we can find a prime r as before, and our claim follows also in this case.

If $1187 \in S$, and 1187 divides $b_{\ell+1}(b_{\ell+1} - 1) \cdots (b_{2\ell-1} + 1)$, then (again with a simple computer program) we could always find a prime t also dividing this product on a power less than ℓ, with the following property: for any a with $p > a \geq t$ there exists a prime r outside $S \cup \{2, 3, 17, 419, 1187\}$ such that $a > r > a/2$. This shows that $k > 2\ell$, and $b_{2\ell+1} \geq t$. Further, for any such $b_{2\ell+1}$ taking the appropriate prime r we conclude that $k \geq 3\ell$ must hold also in this case. For example, if $p = 2377$, $q = 2371$, and $b_{\ell+1} = 2376$, $b_{2\ell} = 2373$, then the primes

$$5, 7, 11, 19, 113, 1187.$$

(possibly together with $2, 3$) divide the ℓ-th power free part of

$$b_{\ell+1}! \cdots b_{2\ell}! \ .$$

So our claim follows with $t = 113$.

I.2) If $p = 1193 > b_{\ell+1}$, then $b_{\ell+1}, \ldots, b_{2\ell}$ must be between p and $q = 1181$, which is the largest prime less than 1187. Now again, a simple check as before assures the statement.

II) Assume now that $b_\ell = n - 1$. Then we have that the ℓ-th power free part of $b_1! \cdots b_\ell!$ is $17^u \times 419^u$ for some u with $1 \leq u \leq \ell - 1$. Now the same procedure as before can be used, with 1187 replaced by 419, and our claim follows. □

Proof of Theorem 3.3. Assume that $n \in F_{\ell_0+1}^{(\ell_0)}$ for some $\ell_0 > \frac{n(\log n)^2}{(\log 2)^2}$. If $n \in D_{\ell_0}^{(\ell_0)}$, then n is an ℓ_0-th power by Theorem 2.1. However, as $n-1$ is a prime, this yields that $n = 2^{\ell_0}$ should be valid, contradicting the assumption made for ℓ_0. Thus $n \in D_{\ell_0+1}^{(\ell_0)}$, whence

$$(n!)^{h_1}(a_2!)^{h_2} \cdots (a_k!)^{h_k}$$

is an ℓ_0-th power with some positive integers $h_1, \ldots, h_k$ and $a_2, \ldots, a_k$ with $\ell_0 > h_i$ $(i = 1, \ldots, k)$, $h_1 + \cdots + h_k = \ell_0 + 1$ and $n > a_2 > \cdots > a_k > 1$. Recalling that $n - 1$ is a prime, this implies that $a_2 = n - 1$, $h_1 + h_2 = \ell_0$ and $k = 3$, $h_3 = 1$. Thus in fact $n^h a!$ is an ℓ_0-th power for some h, a with $1 \leq h < \ell_0$ and $1 < a < n - 1$. This gives

$$h \cdot \mathrm{ord}_p(n) + \mathrm{ord}_p(a!) \equiv 0 \pmod{\ell_0} \tag{6}$$

for each prime p with $p \mid n$. That is,

$$h \equiv -\mathrm{ord}_p(a!)/\mathrm{ord}_p(n) \pmod{\ell_0},$$

for each prime p with $p \mid n$. Observe that we have $\ell_0 > \mathrm{ord}_p(n)$ and by $\log n! \leq n \log n$ also $\ell_0 > \mathrm{ord}_p(a!)$ for any $p \mid n$; even $\ell_0 > \mathrm{ord}_p(a!)\mathrm{ord}_q(n)$ for $p, q \mid n$. So for all primes p, q dividing n we get

$$\mathrm{ord}_p(a!)\mathrm{ord}_q(n) \equiv \mathrm{ord}_q(a!)\mathrm{ord}_p(n) \pmod{\ell_0}.$$

Hence noting that both sides of the above congruence are positive and smaller than ℓ_0, we conclude

$$\mathrm{ord}_p(a!)\mathrm{ord}_q(n) = \mathrm{ord}_q(a!)\mathrm{ord}_p(n) \quad \text{for all } p, q \mid n. \tag{7}$$

Now let ℓ be arbitrary. If $\ell \mid \mathrm{ord}_p(n)$ for all $p \mid n$, then n is an ℓ-th power, and by Theorem 2.1 we have $n \in D_\ell^{(\ell)} \subseteq F_{\ell+1}^{(\ell)}$. So we may assume that $\ell \nmid \mathrm{ord}_p(n)$ for some $p \mid n$. Then for this p we can find a unique h with $1 \leq h < \ell$ satisfying (6) with ℓ_0 replaced by ℓ. Then by (7), the congruence (6) with ℓ in place of ℓ_0 holds for all $q \mid n$ for which $\ell \nmid \mathrm{ord}_q(n)$ with the same h. On the other hand, by (7) again, if $\ell \mid \mathrm{ord}_q(n)$ for some $q \mid n$, since $\ell \nmid \mathrm{ord}_p(n)$ we have $\ell \mid \mathrm{ord}_q(a!)$, which gives again (6) with p replaced by q and ℓ_0 replaced by ℓ. Hence we get that with this h, (6) holds with ℓ_0 replaced by ℓ, for all $p \mid n$. From this the statement follows. □

Proof of Theorem 4.1. For $n < 9$, either n or $n-1$ is a prime. For $n = 9$ we have that $9!8!7!4!$ is a cube, and for $n = 10$ we have that $10!8!7!6!5!3!$ is a cube. So we may assume that $n > 10$.

Let n be such that neither n nor $n-1$ is a prime. Put $n = a_1$, and let $p = P(a_1!)$. Clearly, $\mathrm{ord}_p(a_1!) = 1$. Take $a_2 = p+1$ and $a_3 = p$. Put $q = P(Q_3(a_1!a_2!a_3!))$. If $q = 1$, we are finished. Otherwise q is a prime with $q \leq p-2$. If $\mathrm{ord}_q(Q_3(a_1!a_2!a_3!)) = 1$ then put $a_4 = q+1$ and $a_5 = q$, and if $\mathrm{ord}_q(Q_3(a_1!a_2!a_3!)) = 2$ then let $a_4 = q$. Clearly, in this way we have $a_1 > a_2 > a_3 > a_4 (> a_5)$. Continuing this procedure, we can find $n = a_1 > a_2 > \cdots > a_s > 10$ such that $P(Q_3(a_1!a_2! \cdots a_s!)) < 11$. Now an exhaustive search shows that for any i_2, i_3, i_5, i_7 with $i_j \in \{0,1,2\}$ $(j = 2,3,5,7)$ we can find a subset H of $\{2,3,\ldots,10\}$ such that

$$\mathrm{ord}_j\left(\prod_{h \in H} h!\right) \equiv i_j \pmod 3 \text{ for } j = 2,3,5,7.$$

Hence the theorem follows. □

Proof of Theorem 4.2. Take $n = 2^{6s}$ with $s \geq 1$. Then we have

$$Q_3(2^{6s}!(2^{6s}-1)!(2^{6s}-2)!) = Q_3(2^{6s}(2^{6s}-1)^2) =$$
$$= Q_3(2^{6s}(2^{3s}+1)^2(2^{3s}-1)^2).$$

We also have

$$Q_3((2^{3s}+1)!(2^{3s}-2)!(2^{3s}-3)!) = Q_3((2^{3s}+1)2^{3s}(2^{3s}-1)(2^{3s}-2)^2).$$

Hence

$$Q_3(2^{6s}!(2^{6s}-1)!(2^{6s}-2)!(2^{3s}+1)!(2^{3s}-2)!(2^{3s}-3)!) = Q_3(2^2(2^{3s-1}-1)^2).$$

Further, we have

$$Q_3((2^{3s-1}-1)!(2^{3s-1}-2)!(2^{3s-1}-3)!) = Q_3((2^{3s-1}-1)(2^{3s-1}-2)^2).$$

Combining the last two assertions, we get

$$(8) \quad Q_3(2^{6s}!(2^{6s}-1)!(2^{3s}-2)!(2^{3s}+1)!(2^{3s}-2)!(2^{3s}-3)!\times \\ \times (2^{3s-1}-1)!(2^{3s-1}-2)!(2^{3s-1}-3)!) = Q_3(2^4(2^{3s-2}-1)^2).$$

Hence proceeding by induction, we obtain

$$(9) \quad Q_3(2^{6s}!(2^{6s}-1)!(2^{6s}-2)!(2^{3s}+1)!(2^{3s}-2)!(2^{3s}-3)!\times \\ \times \prod_{i=1}^{3s-3}\{(2^{3s-i}-1)!(2^{3s-i}-2)!(2^{3s-i}-3)!)\} = Q_3(2^{6s-4}\cdot 9).$$

Now multiplying by 3!, the statement follows. The number of factorials is less than $9s$. □

Remark. As an example, we mention that for $s=1$ we obtain that $64!\cdot 63!\cdot 62!\cdot 9!\cdot 6!\cdot 5!\cdot 3!$ is a cube. We also note that as one can easily check, if we start with 2^{2s} in place of 2^{6s} in the proof of Theorem 4.2, then by a similar argument, with some adjustments at the end of the proof (using $2!, 3!, 4!$) we get a similar conclusion.

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A. Bérczes, L. Hajdu
Institute of Mathematics
University of Debrecen
H-4010 Debrecen, P.O. Box 12, Hungary
E-mail address: berczesa@science.unideb.hu
E-mail address: hajdul@science.unideb.hu

A. Dujella
Department of Mathematics
Faculty of Science
University of Zagreb
Bijenička cesta 30, 10000 Zagreb, Croatia
E-mail address: `duje@math.hr`

N. Saradha
INSA Senior Scientist
DAE - Center for Excellence in Basic Sciences
Mumbai University
Mumbai, India
E-mail address: `saradha54@gmail.com`

R. Tijdeman
Mathematical Institute
Leiden University
Postbus 9512, 2300 RA Leiden, The Netherlands
E-mail address: `tijdeman@math.leidenuniv.nl`