On a conjecture regarding balancing with powers of Fibonacci numbers

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To the memory of John Selfridge

Abstract

Here, we show that $(k, \ell, n, r) = (8, 2, 4, 3)$ is the only solution in positive integers of the Diophantine equation

$$F_1^k + F_2^k + \cdots + F_{n-1}^k = F_{n+1}^\ell + \cdots + F_{n+r}^\ell,$$

where F_m is the mth Fibonacci number.

1. Introduction

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by $F_0 = 0,\ F_1 = 1$ and

$$F_{n+2} = F_{n+1} + F_n \qquad \text{for all} \quad n \geq 0. \tag{1}$$

The following conjecture was proposed in [1].

Conjecture 1. *The only quadruple* (k, ℓ, n, r) *of positive integers satisfying the Diophantine equation*

$$F_1^k + F_2^k + \cdots + F_{n-1}^k = F_{n+1}^\ell + \cdots + F_{n+r}^\ell \tag{2}$$

is $(8, 2, 4, 3)$.

Conjecture 1 is a version involving powers of Fibonacci numbers of the classical problem concerning *balancing numbers*, which are positive integers n such that the equality

$$1 + 2 + \cdots + (n-1) = (n+1) + \cdots + (n+r)$$

holds with some positive integer r. Several variations of this problem have been previously considered in the literature (see [2], [10]).

The authors of [1] also show that every solution of equation (2) has $\ell < k$ and that there is no such solution with $(k, \ell) = (2, 1)$, $(3, 1)$, or $(3, 2)$. In particular, all solutions of equation (2) have $k \geq 4$. Observe also that $n \geq 4$. Here, we confirm Conjecture 1. We record the result as follows.

Theorem 1. *Conjecture 1 holds.*

Our method uses linear forms in logarithms, LLL, and some elementary considerations.

We recall that the formula

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{holds for} \quad n \geq 0, \quad \text{where} \quad (\alpha, \beta) := \left(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2} \right). \tag{3}$$

In particular, the inequality $\alpha^{n-2} \leq F_n \leq \alpha^{n-1}$ holds for all $n \geq 1$. We will use this inequality several times throughout.

2. Preliminary inequalities

Observe that

$$F_1^k + F_2^k + \cdots + F_{n-1}^k > F_{n-1}^k > (\alpha^{n-3})^k = \alpha^{k(n-1)-2k},$$

On the other hand,

$$F_1^k + F_2^k + \cdots + F_{n-1}^k \leq (F_1 + \cdots + F_{n-1})^k = (F_{n+1} - 1)^k < F_{n+1}^k < (\alpha^n)^k = \alpha^{nk},$$

where we used the known fact that the identity $F_1 + F_2 + \cdots + F_m = F_{m+2} - 1$ holds for all $m \geq 1$. Thus,

$$\alpha^{k(n-1)-2k} < F_1^k + F_2^k + \cdots + F_{n-1}^k < \alpha^{k(n-1)+k}. \tag{4}$$

In a similar way, we also get that

$$\alpha^{\ell(n+r)-2\ell} < F_{n+1}^{\ell} + \cdots + F_{n+r}^{\ell} < \alpha^{\ell(n+r)+\ell}. \tag{5}$$

Comparing bounds (4) and (5), we get

$$k(n-1) - 2k < \ell(n+r) + \ell \qquad \text{and} \qquad \ell(n+r) - 2\ell < k(n-1) + k,$$

so

$$|k(n-1) - \ell(n+r)| < \max\{2k+\ell, k+2\ell\} = 2k + \ell. \tag{6}$$

We record this as a lemma.

Lemma 2. *Any positive integer solution* (k, ℓ, n, r) *of equation* (2) *satisfies inequality* (6).

3. Initial bounds on k and ℓ

Write

$$\begin{aligned} N &:= F_{n+1}^{\ell} + \cdots + F_{n+r}^{\ell} \\ &= F_{n+r}^{\ell}\left(\left(\frac{F_{n+1}}{F_{n+r}}\right)^{\ell} + \left(\frac{F_{n+2}}{F_{n+r}}\right)^{\ell} + \cdots + \left(\frac{F_{n+r-1}}{F_{n+r}}\right)^{\ell} + 1\right) \\ &=: F_{n+r}^{\ell}(1+S). \end{aligned} \tag{7}$$

Observe that $S \geq 0$. The inequality

$$\frac{F_{m-1}}{F_m} \leq \frac{2}{3} \qquad \text{holds for all} \quad m \geq 3. \tag{8}$$

Indeed, the above inequality is equivalent to $2F_m \geq 3F_{m-1}$, or $2(F_{m-1} + F_{m-2}) \geq 3F_{m-1}$, or $2F_{m-2} \geq F_{m-1} = F_{m-2} + F_{m-3}$, or $F_{m-2} \geq F_{m-3}$, which is indeed true for all $m \geq 3$. Hence,

$$\frac{F_{n+r-1}}{F_{n+r}} \leq \frac{2}{3}, \quad \frac{F_{n+r-2}}{F_{n+r}} = \frac{F_{n+r-2}}{F_{n+r-1}} \cdot \frac{F_{n+r-1}}{F_{n+r}} \leq \left(\frac{2}{3}\right)^2, \quad \ldots, \quad \frac{F_{n+1}}{F_{n+r}} < \left(\frac{2}{3}\right)^{r-1}. \tag{9}$$

Thus,

$$S = \left(\frac{F_{n+1}}{F_{n+r}}\right)^{\ell} + \cdots + \left(\frac{F_{n+r-1}}{F_{n+r}}\right)^{\ell} < \sum_{j\geq 1}\left(\frac{2}{3}\right)^{j\ell} = \left(\frac{2}{3}\right)^{\ell}\frac{1}{1-(2/3)^{\ell}} \leq \frac{3}{1.5^{\ell}}, \tag{10}$$

because $\ell \geq 1$. In a similar way, we write

$$\begin{aligned} N &= F_1^k + \cdots + F_{n-1}^k \\ &= F_{n-1}^k\left(\left(\frac{F_1}{F_{n-1}}\right)^k + \left(\frac{F_2}{F_{n-1}}\right)^k + \cdots + \left(\frac{F_{n-2}}{F_{n-1}}\right)^k + 1\right) \\ &=: F_{n-1}^k(1+S_1). \end{aligned} \tag{11}$$

The argument which proved inequality (9) based on inequality (8) shows that

$$\frac{F_{n-2}}{F_{n-1}} \le \frac{2}{3}, \quad \frac{F_{n-3}}{F_{n-1}} = \frac{F_{n-3}}{F_{n-2}} \cdot \frac{F_{n-2}}{F_{n-1}} \le \left(\frac{2}{3}\right)^2, \quad \ldots, \quad \frac{F_2}{F_{n-1}} < \left(\frac{2}{3}\right)^{n-3}.$$

Furthermore,

$$\frac{F_1}{F_{n-1}} = \frac{F_2}{F_{n-1}} \le \left(\frac{2}{3}\right)^{n-3}.$$

Thus,

$$\begin{aligned} S_1 &= \left(\frac{F_1}{F_{n-1}}\right)^k + \cdots + \left(\frac{F_{n-2}}{F_{n-1}}\right)^k < \left(\frac{2}{3}\right)^{k(n-3)} + \sum_{j\ge 1}\left(\frac{2}{3}\right)^{jk} \\ &\le \left(\frac{2}{3}\right)^k \left(1 + \frac{1}{1-(2/3)^k}\right) \le \frac{3}{1.5^k}, \end{aligned} \tag{12}$$

where for the last inequality we used the fact that $k \ge 4$.

Since

$$N = F_{n-1}^k(1+S_1) = F_{n+r}^{\ell}(1+S),$$

we get that

$$|F_{n-1}^k - F_{n+r}^{\ell}| = |F_{n-1}^k S_1 - F_{n+r}^{\ell} S| < M \max\{S, S_1\}, \tag{13}$$

where $M := \max\{F_{n-1}^k, F_{n+r}^{\ell}\}$. Dividing both sides of the above inequality by M, we get

$$|F_{n-1}^{\varepsilon k} F_{n+r}^{-\varepsilon \ell} - 1| < \frac{3}{1.5^{\ell}}, \tag{14}$$

where $\varepsilon = 1$ or -1, according to whether $M = F_{n+r}^{\ell}$ or F_{n-1}^k, respectively.

We shall use several times a result of Matveev (see [9], or Theorem 9.4 in [3]), which asserts that if $\alpha_1, \alpha_2, \ldots, \alpha_K$ are positive real algebraic numbers in an algebraic number field $\mathbb{K}$ of degree D, $b_1, b_2, \ldots, b_K$ are rational integers, and

$$\Lambda := \alpha_1^{b_1}\alpha_2^{b_2}\cdots\alpha_K^{b_K} - 1$$

is not zero, then

$$|\Lambda| > \exp\left(-1.4 \times 30^{K+3} \times K^{4.5} D^2 (1+\log D)(1+\log B) A_1 A_2 \cdots A_K\right), \tag{15}$$

where

$$B \ge \max\{|b_1|, |b_2|, \ldots, |b_K|\},$$

and

$$A_i \ge \max\{D h(\alpha_i), |\log \alpha_i|, 0.16\}, \qquad \text{for all} \quad i = 1, 2, \ldots, K. \tag{16}$$

Here, for an algebraic number η we write $h(\eta)$ for its logarithmic height whose formula is

$$h(\eta) := \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log\left(\max\{|\eta^{(i)}|, 1\}\right)\right),$$

with d being the degree of η over $\mathbb{Q}$ and

$$f(X) := a_0 \prod_{i=1}^{d} (X - \eta^{(i)}) \in \mathbb{Z}[X]$$

being the minimal primitive polynomial over the integers having positive leading coefficient a_0 and η as a root.

In a first application of Matveev's theorem, we take $K := 2,\ \alpha_1 := F_{n-1},\ \alpha_2 := F_{n+r}$. We also take $b_1 := \varepsilon k$, and $b_2 := -\varepsilon \ell$. Thus,

$$\Lambda_1 := F_{n-1}^{\varepsilon k} F_{n+r}^{-\varepsilon \ell} - 1 \tag{17}$$

is the expression appearing under the absolute value in the left–hand side of inequality (14). Let us check that $\Lambda_1 \neq 0$. If $\Lambda_1 = 0$, it follows that $F_{n-1}^{k} = F_{n+r}^{\ell}$. Hence, F_{n-1} and F_{n+k} are multiplicatively dependent. However, Carmichael's Primitive Divisor Theorem (see [4]) asserts that that if $n > 12$, then F_n has a primitive prime factor; that is, a prime factor p such that p does not divide F_m for any $m < n$. In particular, if $n + r > 12$, then F_{n-1} and F_{n+r} are multiplicatively independent. A quick look at the remaining cases shows that the only instance in which F_{n-1} and F_{n+r} are multiplicatively dependent is when $F_{n-1} = 2$ and $F_{n+r} = 8$, so $n = 4$ and $r = 2$. But in this case, equation (2) is

$$1^k + 1^k + 2^k = 5^\ell + 8^\ell,$$

which has no solutions anyway since its left–hand side is even and its right–hand side is odd. Hence, indeed $\Lambda_1 \neq 0$.

Since $\ell < k$, it follows that $B = k$. Since α_1 and α_2 are rational numbers, it follows that we can take $D := 1$. Next, since the inequality $F_m < \alpha^m$ holds for all positive integers m, we can take $A_1 := (n-1)\log\alpha$ and $A_2 := (n+r)\log\alpha$, and then inequalities (16) hold for both $i = 1,\ 2$. Now Matveev's theorem tells us that

$$|\Lambda_1| > \exp\left(-C_1 \times (n-1)\log\alpha \times (n+r)\log\alpha \times (1 + \log k)\right), \tag{18}$$

where

$$C_1 := 1.4 \times 30^5 \times 2^{4.5} < 8 \times 10^8.$$

Taking logarithms in inequality (14) and comparing the resulting inequality with (18), we get

$$-C_1(\log\alpha)^2(n-1)(n+r)(1+\log k) < \log|\Lambda_1| < -\ell\log(1.5) + \log 3,$$

so

$$\ell - \frac{\log 3}{\log(1.5)} < \frac{C_1(\log\alpha)^2}{\log(1.5)}(n-1)(n+r)(1+\log k), \tag{19}$$

which leads to

$$\ell < 5 \times 10^8 n(n+r)(1+\log k) < 10^9 n(n+r)\log k, \tag{20}$$

because $\log k \geq \log 4 > 1$.

Recall now that, by Lemma 2 and the fact that $n \geq 4$, we have

$$3k \leq (n-1)k \leq (n+r)\ell + 2k + \ell, \qquad \text{therefore} \qquad k \leq \ell(n+r+1). \tag{21}$$

Thus,

$$\ell < 10^9 n(n+r)\log(\ell(n+r+1)). \tag{22}$$

If $\ell \leq n+r$, then we have an inequality which is better than inequality (22). Otherwise, $\ell \geq n+r+1$, therefore

$$\ell \leq 2 \times 10^9 n(n+r)\log \ell,$$

so

$$\frac{\ell}{\log \ell} < 2 \times 10^9 n(n+r). \tag{23}$$

It is well-known and easy to prove that if $A \geq 3$ and $x/\log x < A$, then $x < 2A\log A$ (see, for example, [7]). Thus, taking $A := 2 \times 10^9 n(n+r)$, inequality (23) gives us

$$\begin{aligned} \ell &< 2(2 \times 10^9 n(n+r))\log(2 \times 10^9 n(n+r)) \\ &< 4 \times 10^9 n(n+r)(\log(2 \times 10^9) + 2\log(n+r)) \\ &< 4 \times 10^9 n(n+r)(22 + 2\log(n+r)) \\ &< 4 \times 10^9 n(n+r)(16\log(n+r)) \\ &< 6.5 \times 10^{10} n(n+r)\log(n+r). \end{aligned}$$

In the above chain of inequalities, we used that fact that $n+r \geq 5$, which implies that $\log(n+r) \geq \log 5 = 1.60944\ldots > 22/14 = 1.57143\ldots$. Thus,

$$\ell < 6.5 \times 10^{10} n(n+r)\log(n+r). \tag{24}$$

From estimate (21), we also deduce that

$$\begin{aligned} k &< \ell(n+r+1) < 6.5 \times 10^{10} n(n+r)(n+r+1)\log(n+r) \\ &< 8 \times 10^{10} n(n+r)^2 \log(n+r), \end{aligned} \tag{25}$$

where we used the fact that $n+r \geq 5$, which implies that $(n+r+1)/(n+r) \leq 6/5$. We record what we have just proved.

Lemma 3. *If (k, ℓ, n, r) is a solution in positive integers of equation (2), then both inequalities*

$$\begin{aligned} \ell &< 7 \times 10^{10} n(n+r)\log(n+r), \\ k &\leq \ell(n+r+1) < 8 \times 10^{10} n(n+r)^2 \log(n+r) \end{aligned}$$

hold.

4. The case of small n and r

Here, we assume that $n \leq 3000,\ r \leq 3000$. Thus, by Lemma 3, we have

$$\ell < 7 \times 10^{10} \times 3000 \times 6000 \times \log 6000 < 1.1 \times 10^{19}, \quad k \leq \ell(n+r+1) < 6.6 \times 10^{23}.$$

Now put $\Gamma_1 := k \log F_{n-1} - \ell \log F_{n+r}$. Inequality (14) tells us that

$$|e^{-|\Gamma_1|} - 1| = |\Lambda_1| < \frac{3}{1.5^\ell}.$$

Assuming that $\ell \geq 5$, we then have that $3/1.5^\ell < 1/2$, so that $|e^{-|\Gamma_1|} - 1| < 1/2$. This leads to $e^{|\Gamma_1|} < 2$, therefore

$$|\Gamma_1| < e^{|\Gamma_1|}|e^{-|\Gamma_1|} - 1| < \frac{6}{1.5^\ell}.$$

Dividing the last inequality above by $\ell \log F_{n-1}$, we get that

$$\left|\frac{\log F_{n+r}}{\log F_{n-1}} - \frac{k}{\ell}\right| < \frac{6}{\ell(\log F_{n-1})1.5^\ell} \leq \frac{6}{\ell(\log 2)1.5^\ell}.$$

The left–hand side above is $< 1/(2\ell^2)$ for all $\ell \geq 14$. Thus, by a criterion of Legendre, it follows that if $\ell \geq 14$, then k/ℓ is a convergent of the continued fraction of the number $\gamma := (\log F_{n+r})/(\log F_{n-1})$. Hence, $k/\ell = p_i/q_i$, where p_i/q_i is the ith convergent of γ and furthermore $q_i < 1.1 \times 10^{19}$. This gives a certain number of possibilities for the ratio k/ℓ once n and r are fixed. Fixing the ratio $k/\ell = \kappa/\lambda$ with coprime positive integers κ and λ, we can write $k = \kappa d$ and $\ell = \lambda d$ for some positive integer d, which is the greatest common divisor of k and ℓ. Then, again with $n,\ r$ fixed and κ and λ fixed also, inequality (14) gives

$$|(F_{n-1}^{\varepsilon\kappa} F_{n+r}^{-\varepsilon\lambda})^d - 1| < \frac{3}{(1.5^\lambda)^d},$$

which gives a few possibilities for d. Hence, when n and r are fixed, we get a certain number of possibilities for the pair $(k, \ell) = (\kappa d, \lambda d)$. All this was when $\ell \geq 14$, but if $\ell \leq 13$ and n and r are fixed, then we have only a few possibilities for k as well. Then we test all such possible quadruples (k, ℓ, n, r) and check whether equation (2) is satisfied. This computation took some 20 hours and revealed no additional solutions (k, ℓ, n, r) to equation (2) aside from $(8, 2, 4, 3)$.

We record what we have obtained as follows.

Lemma 4. *If* (k, ℓ, n, r) *is a positive integer solution to equation* (2) *other than* $(8, 2, 4, 3)$*, then* $\max\{n, r\} \geq 3001$.

5. A bound for r in terms of n

From now on, we assume that $\max\{n, r\} \geq 3001$. We look at the right–hand side of (2) more closely. Recall that

$$N := F_{n+1}^\ell + \cdots + F_{n+r}^\ell$$

(see (7)). We let $t := \max\{n, \lfloor (n+r)/2 \rfloor\}$ and put

$$N_1 := F_{n+1}^{\ell} + \cdots + F_t^{\ell}.$$

Observe that $N_1 = 0$ if $r \le n+1$. If $N_1 \neq 0$, then $r \ge n+2$, therefore we have that $t = \lfloor (n+r)/2 \rfloor \le (n+r)/2$. Thus,

$$\begin{aligned} N_1 &= F_t^{\ell} \left(\left(\frac{F_{n+1}}{F_t} \right)^{\ell} + \left(\frac{F_{n+2}}{F_t} \right)^{\ell} + \cdots + \left(\frac{F_{t-1}}{F_t} \right)^{\ell} + 1 \right) \\ &< F_t^{\ell} \left(\frac{3}{1.5^{\ell}} + 1 \right) \le 3F_t^{\ell} < 3(F_{2t}^{\ell})^{1/2} \le 3(F_{n+r}^{\ell})^{1/2} \le 3N^{1/2}. \end{aligned} \tag{26}$$

In the above estimates, we invoked the argument used at (10) to bound S, as well as the known fact that the inequality $F_{2m} \ge F_m^2$ holds for all positive integers m. Thus, the inequality $N_1 \le 3\sqrt{N}$ holds regardless of whether N_1 is zero or not. Before moving further, observe that the inequality

$$\sqrt{N} < \frac{\alpha N}{\alpha^{(n+r)/2}} \tag{27}$$

holds, because this inequality is equivalent to $N \ge \alpha^{n+r-2}$, which holds since

$$N \ge F_{n+r}^{\ell} \ge F_{n+r} > \alpha^{n+r-2}.$$

Hence, using (26) and (27), we get

$$N_1 < 3\sqrt{N} < \frac{3\alpha N}{\alpha^{(n+r)/2}} < \frac{5N}{\alpha^{(n+r)/2}}. \tag{28}$$

We next look at

$$N_2 := N - N_1 = F_{t+1}^{\ell} + \cdots + F_{n+r}^{\ell}.$$

Let $j \in [t+1, n+r]$. Write

$$F_j^{\ell} = \left(\frac{\alpha^j - \beta^j}{5^{1/2}} \right)^{\ell} = \frac{\alpha^{j\ell}}{5^{\ell/2}} \left(1 - \frac{(-1)^j}{\alpha^{2j}} \right)^{\ell}. \tag{29}$$

Observe that, by Lemma 3, we have that

$$\frac{\ell}{\alpha^{2j}} < \frac{7 \times 10^{10} (n+r)^2 \log(n+r)}{\alpha^{n+r}} < \frac{1}{\alpha^{(n+r)/2}}.$$

The last inequality holds whenever $n+r \ge 153$, which is the case for us. Since $n+r \ge 3002$, the right–hand side above is $< \alpha^{-1500} < 10^{-300}$. If j is odd, then

$$\begin{aligned} 1 < \left(1 - \frac{(-1)^j}{\alpha^{2j}} \right)^{\ell} &= \left(1 + \frac{1}{\alpha^{2j}} \right)^{\ell} = \exp\left(\ell \log \left(1 + \frac{1}{\alpha^{2j}} \right) \right) < \exp\left(\frac{\ell}{\alpha^{2j}} \right) \\ &< \exp\left(\frac{1}{\alpha^{(n+r)/2}} \right) < 1 + \frac{2}{\alpha^{(n+r)/2}}, \end{aligned} \tag{30}$$

because the argument inside the exponential is $< 10^{-300}$. Similarly, if j is even, then

$$\begin{aligned} 1 > \left(1-\frac{(-1)^j}{\alpha^{2j}}\right)^\ell & = \left(1-\frac{1}{\alpha^{2j}}\right)^\ell = \exp\left(\ell\log\left(1-\frac{1}{\alpha^{2j}}\right)\right) \\ & > \exp\left(-\frac{2\ell}{\alpha^{2j}}\right) > \exp\left(-\frac{2}{\alpha^{(n+r)/2}}\right) > 1-\frac{2}{\alpha^{(n+r)/2}}. \end{aligned} \tag{31}$$

Formula (29), together with bounds (30) and (31), gives

$$\left|F_j^\ell - \frac{\alpha^{j\ell}}{5^{\ell/2}}\right| < \frac{\alpha^{j\ell}}{5^{\ell/2}}\left|\left(1-\frac{(-1)^j}{\alpha^{2j}}\right)^\ell - 1\right| < \left(\frac{\alpha^{j\ell}}{5^{\ell/2}}\right)\left(\frac{2}{\alpha^{(n+r)/2}}\right). \tag{32}$$

Put

$$x := \frac{1}{\alpha^{(n+r)/2}}.$$

Since $x < 10^{-300}$, inequality (32) certainly implies that $\alpha^{j\ell}/5^{\ell/2} < 1.5F_j^\ell$, therefore

$$\left|F_j^\ell - \frac{\alpha^{j\ell}}{5^{\ell/2}}\right| < 2x\left(\frac{\alpha^{j\ell}}{5^{\ell/2}}\right) < 3xF_j^\ell. \tag{33}$$

The above inequality applied for $j = t+1, \ldots, n+r$, gives immediately that if we put

$$N_3 := \sum_{j=t+1}^{n+r} \frac{\alpha^{j\ell}}{5^{\ell/2}},$$

then the inequality

$$|N_2 - N_3| < 3x\left(F_{t+1}^\ell + \cdots + F_{n+r}^\ell\right) = 3xN_2 \le 3xN = \frac{3N}{\alpha^{(n+r)/2}} \tag{34}$$

holds. It remains to estimate N_3. Observe first that

$$N_3 = \frac{\alpha^{(n+r+1)\ell} - \alpha^{(t+1)\ell}}{5^{\ell/2}(\alpha^\ell - 1)}. \tag{35}$$

In particular,

$$\left|N - \frac{\alpha^{(t+1)\ell}(\alpha^{(n+r-t)\ell} - 1)}{5^{\ell/2}(\alpha^\ell - 1)}\right| = |N - N_3| \le N_1 + |N_2 - N_3| < \frac{8N}{\alpha^{(n+r)/2}}, \tag{36}$$

by estimates (28) and (34). Now, by estimate (33) for $j = t+1$, we have

$$\frac{\alpha^{(t+1)\ell}}{5^{\ell/2}} < 2F_{t+1}^\ell \le 2\left(\frac{F_{n+r}}{F_{n+r-t}}\right)^\ell \le \frac{2N}{F_{n+r-t}^\ell} \le \frac{2N}{\alpha^{(n+r-t-2)\ell}},$$

where we used the fact that for positive integers a and b the inequality $F_{a+b} \ge F_aF_{b+1}$ holds (with $a := n+r-t$ and $b := t$). Hence,

$$\frac{\alpha^{(t+1)\ell}}{5^{\ell/2}(\alpha^\ell - 1)} < \frac{2N}{(\alpha^\ell - 1)\alpha^{(n+r-t-2)\ell}} < \frac{6N}{\alpha^{(n+r-t-1)\ell}}, \tag{37}$$

where we used the fact that the inequality $\alpha^\ell/(\alpha^\ell - 1) < 2.62 < 3$ holds for all positive integers ℓ. Thus, from formula (35) and estimates (36) and (37), we get that

$$\left|N - \frac{\alpha^{(n+r+1)\ell}}{5^{\ell/2}(\alpha^\ell - 1)}\right| \le |N - N_3| + \frac{\alpha^{(t+1)\ell}}{5^{\ell/2}(\alpha^\ell - 1)} < 8N\left(\frac{1}{\alpha^{(n+r)/2}} + \frac{1}{\alpha^{(n+r-t-1)\ell}}\right), \tag{38}$$

by (36). The above inequality (38) implies that

$$\begin{aligned}\left|N(\alpha^\ell - 1) - \frac{\alpha^{(n+r+1)\ell}}{5^{\ell/2}}\right| &< 8N(\alpha^\ell - 1)\left(\frac{1}{\alpha^{(n+r)/2}} + \frac{1}{\alpha^{(n+r-t-1)\ell}}\right)\\ &< 8N\left(\frac{1}{\alpha^{(n+r)/2-\ell}} + \frac{1}{\alpha^{(n+r-t-2)\ell}}\right).\end{aligned}$$

Hence,

$$\left|N\alpha^\ell - \frac{\alpha^{(n+r+1)\ell}}{5^{\ell/2}}\right| < N\left(1 + \frac{8}{\alpha^{(n+r)/2-\ell}} + \frac{8}{\alpha^{(n+r-t-2)\ell}}\right),$$

so that

$$\left|N - \frac{\alpha^{(n+r)\ell}}{5^{\ell/2}}\right| < 8N\left(\frac{1}{\alpha^\ell} + \frac{1}{\alpha^{(n+r)/2}} + \frac{1}{\alpha^{(n+r-t-1)\ell}}\right). \tag{39}$$

Let's take a break and see what we have done so far. Let us look at the last term on the right–hand sides both in (38) as well as in (39) above.

If $t = \lfloor (n+r)/2 \rfloor$, then $r \ge n$, and

$$(n+r-t-1)\ell \ge (n+r-\lfloor (n+r)/2 \rfloor - 1)\ell \ge ((n+r)/2 - 1)\ell,$$

so the third term on the right–hand side of (39) is majorized by the second term because $n + r \ge 3002$.

If $t = n$, then $(n+r-t-1)\ell = (r-1)\ell \ge r\ell/2$ if $r \ge 2$.

However, if $r = 1$, we then get $n + r - t - 1 = 0$, so the third term on the right–hand side of both inequalities (38) and (39) is too large to be useful.

So, let us take a closer look at the case $r = 1$. In this case, we simply have

$$N = F_{n+1}^\ell.$$

Hence,

$$\left|N - \frac{\alpha^{(n+1)\ell}}{5^{\ell/2}}\right| < 3xF_{n+1}^\ell = 3xN = \frac{3N}{\alpha^{(n+r)/2}}, \tag{40}$$

by estimate (33) with $j = n + 1 = n + r$. Furthermore, the above estimate (40) implies that

$$\left|\frac{N\alpha^\ell}{\alpha^\ell - 1} - \frac{\alpha^{(n+2)\ell}}{5^{\ell/2}(\alpha^\ell - 1)}\right| < \left(\frac{\alpha^\ell}{\alpha^\ell - 1}\right)\left(\frac{3N}{\alpha^{(n+1)/2}}\right) < \frac{8N}{\alpha^{(n+1)/2}}.$$

Hence,

$$\left|N - \frac{\alpha^{(n+2)\ell}}{5^{\ell/2}(\alpha^\ell - 1)}\right| < \frac{N}{\alpha^\ell - 1} + \frac{8N}{\alpha^{(n+1)/2}} < 8N\left(\frac{1}{\alpha^\ell} + \frac{1}{\alpha^{(n+1)/2}}\right)$$
$$= 8N\left(\frac{1}{\alpha^{(n+r)/2}} + \frac{1}{\alpha^{(n+r-t)\ell/2}}\right). \quad (41)$$

Comparing estimates (38) and (41), as well as (39) and (40), we get that estimates (38) and (39) hold also when $r = 1$ with the exponent of α on the last term in the right–hand side replaced by $(n + r - t)/2$ (instead of $n + r - t - 1$). Since $n + r - t - 1 \geq (n + r - t)/2$ holds whenever $r > 1$, we can record our conclusion as follows:

Lemma 5. *Let* (k, ℓ, n, r) *be a solution other than* $(8, 2, 4, 3)$ *of equation* (2)*. Putting*

$$N := F_{n+1}^\ell + \cdots + F_{n+r}^\ell,$$

and $t := \max\{n, \lfloor (n + r)/2 \rfloor\}$*, then all three inequalities*

$$\left|N - \frac{\alpha^{(n+r)\ell}}{5^{\ell/2}}\right| < 8N\left(\frac{1}{\alpha^\ell} + \frac{1}{\alpha^{(n+r)/2}} + \frac{1}{\alpha^{(n+r-t)\ell/2}}\right), \quad (42)$$

$$\left|N - \frac{\alpha^{(n+r+1)\ell}}{5^{\ell/2}(\alpha^\ell - 1)}\right| < 8N\left(\frac{1}{\alpha^{(n+r)/2}} + \frac{1}{\alpha^{(n+r-t)\ell/2}}\right), \quad (43)$$

and

$$\left|N - \frac{\alpha^{(t+1)\ell}(\alpha^{(n+r-t)\ell} - 1)}{5^{\ell/2}(\alpha^\ell - 1)}\right| < \frac{8N}{\alpha^{(n+r)/2}} \quad (44)$$

hold.

Now we return to

$$N = F_1^k + \cdots + F_{n-1}^k = F_{n-1}^k(1 + S_1),$$

where $0 < S_1 < 3/1.5^k$ (see formula (11) and estimate (12)). Since $k \geq 4$, we get that $S_1 < 3/5$, and in particular $5N/8 < F_{n-1}^k < N$. Hence,

$$|N - F_{n-1}^k| < F_{n-1}^k S_1 < \frac{3N}{1.5^k}. \quad (45)$$

Comparing estimates (45), (42) and (43), and using also the fact that $\alpha > 1.5$, we get

$$\left|F_{n-1}^k - \frac{\alpha^{(n+r)\ell}}{5^{\ell/2}}\right| < 8N\left(\frac{1}{1.5^\ell} + \frac{1}{1.5^k} + \frac{1}{1.5^{(n+r)/2}} + \frac{1}{1.5^{(n+r-t)\ell/2}}\right), \quad (46)$$

and

$$\left|F_{n-1}^k - \frac{\alpha^{(n+r+1)\ell}}{5^{\ell/2}(\alpha^\ell - 1)}\right| < 8N\left(\frac{1}{1.5^k} + \frac{1}{1.5^{(n+r)/2}} + \frac{1}{1.5^{(n+r-t)\ell/2}}\right). \quad (47)$$

We divide both sides of equations (46) and (47) by F_{n-1}^k and keep in mind that $N/F_{n-1}^k < 8/5$, to get

$$\left|F_{n-1}^{-k}\alpha^{(n+r)\ell}5^{-\ell/2} - 1\right| < 13\left(\frac{1}{1.5^\ell} + \frac{1}{1.5^k} + \frac{1}{1.5^{(n+r)/2}} + \frac{1}{1.5^{(n+r-t)\ell/2}}\right), \tag{48}$$

and

$$\left|F_{n-1}^{-k}\alpha^{(n+r+1)\ell}5^{-\ell/2}(\alpha^\ell - 1)^{-1} - 1\right| < 13\left(\frac{1}{1.5^k} + \frac{1}{1.5^{(n+r)/2}} + \frac{1}{1.5^{(n+r-t)\ell/2}}\right). \tag{49}$$

Recall that our goal in this section is to bound r. We distinguish several cases.

Case 1. $r \le n$.

In this case, by Lemma 3, we get that

$$\begin{aligned} \ell &\le 7 \times 10^{10} n(2n)\log(2n) = 1.4 \times 10^{11} n^2(\log 2 + \log n) \\ &\le 1.4 \times 10^{11} n^2 \times (3/2) \times \log n < 2.1 \times 10^{11} n^2 \log n, \end{aligned} \tag{50}$$

where we used the fact that $n \ge 4 \ge 2^2$, so that $\log n \ge 2\log 2$. Furthermore,

$$k < \ell(n+r+1) = \ell(n+r)\left(\frac{n+r+1}{n+r}\right) \le \ell(2n)\left(\frac{3003}{3002}\right) < 4.3 \times n^3 \log n, \tag{51}$$

where we used the fact that $n + r \ge 3002$. We also record that

$$r \le n \tag{52}$$

for future referencing.

From now on, we assume that $r > n$. In particular, $t = \lfloor (n+r)/2 \rfloor$, therefore

$$(n+r-t)\ell/2 \ge (n+r)\ell/4. \tag{53}$$

We shall apply Matveev's theorem to bound from below the left–hand sides of (48) and (49). Let us check that they are not zero. If the left–hand side of (48) is zero, then $\alpha^{2(n+r)\ell} = F_{n-1}^{2k}5^\ell \in \mathbb{Z}$, which is impossible since no power of α of positive integer exponent is an integer. If the left–hand side of (49) is zero, we then get that

$$\frac{\alpha^{(n+r+1)\ell}}{\alpha^\ell - 1} = F_{n-1}^k 5^{\ell/2}.$$

Conjugating the above relation in $\mathbb{Q}(\sqrt{5})$ and multiplying the two resulting relations we get

$$\frac{(-1)^{(n+r+1)\ell}}{(\alpha^\ell - 1)(\beta^\ell - 1)} = (-1)^\ell F_{n-1}^{2k}5^\ell.$$

The right–hand side above is an integer of absolute value larger than 1, while the left–hand side above is the reciprocal of an integer. This is a contradiction. Hence, the left–hand sides of (48) and (49) are non-zero.

We start with a lower bound on the left–hand side of inequality (48). For this, we take $K := 3$, $\alpha_1 = F_{n-1}$, $\alpha_2 := \alpha$, $\alpha_3 := \sqrt{5}$. We also take $b_1 := -k$, $b_2 := (n+r)\ell$, $b_3 := \ell$. Hence,

$$\Lambda_2 := \alpha_1^{b_1}\alpha_2^{b_2}\alpha_3^{b_3} - 1 = F_{n-1}^{-k}\alpha^{(n+r)\ell}5^{-\ell/2} - 1$$

is the expression which appears under the absolute value in the left–hand side of inequality (48). We have already checked that $\Lambda_2 \neq 0$. Observe that α_1, α_2, α_3 are all real and belong to the field $\mathbb{K} := \mathbb{Q}(\sqrt{5})$, so we can take $D := 2$. Next, since $F_{n-1} < \alpha^n$, it follows that we can take $A_1 := 2n\log\alpha > D\log F_{n-1} = Dh(\alpha_1)$. Next, since $h(\alpha_2) = (\log\alpha)/2 = 0.240606\ldots$, it follows that we can take $A_2 := 0.5 > Dh(\alpha_2)$. Since $h(\alpha_3) = (\log 5)/2 = 0.804719\ldots$, it follows that we can take $A_3 := 1.61 > Dh(\alpha_3)$. Finally, Lemma 3 tells us that we can take

$$\begin{aligned} B &= 1.3\times 10^{12}r^4 > 8\times 10^{10}(2r)^4 > 8\times 10^{10}(n+r)^4 \\ &> 8\times 10^{10}n(n+r)^2\log(n+r) \geq \ell(n+r+1) \\ &\geq \max\{k, \ell(n+r), \ell\} = \max\{|b_1|, |b_2|, |b_3|\}. \end{aligned}$$

Matveev's theorem tells us that

$$|\Lambda_2| > \exp(-C_2(1+\log B)A_1A_2A_3), \tag{54}$$

where

$$C_2 := 1.4\times 30^6\times 3^{4.5}\times 2^2(1+\log 2) < 10^{12}.$$

Thus,

$$\begin{aligned} C_2(1+\log B)A_1A_2A_3 &< 10^{12}\times(2\log\alpha)\times 0.5\times 1.61 \\ &\times (1+\log(1.3\times 10^{12})+4\log r)n \\ &< 8\times 10^{11}(29+4\log r)n \\ &< 8\times 10^{11}(8\log r)n \\ &< 7\times 10^{12}n\log r, \end{aligned} \tag{55}$$

where we used the fact that $29+4\log r < 8\log r$, because $r \geq 3001$ (otherwise, that is if $r \leq 3000$, then we get that $n < r \leq 3000$ also, which is a case already treated). Now inequalities (15), (48), (53), and (54) show that if we put

$$\lambda_1 := \min\{\ell, k, (n+r)/2, (n+r)\ell/4\}, \tag{56}$$

then the inequality

$$\exp\left(-7\times 10^{12}n\log r\right) < \frac{60}{1.5^{\lambda_1}}$$

holds. This in term yields

$$\lambda_1 < \frac{\log 60}{\log 1.5} + 7\times 10^{12}(\log 1.5)^{-1}n\log r < 2\times 10^{13}n\log r. \tag{57}$$

We already know that $\ell < k$. We distinguish the following cases.

Case 2. $\lambda_1 \in \{(n+r)/2, (n+r)\ell/4\}$.

Inequality (57) gives in this case that

$$r/4 < \lambda_1 < 2 \times 10^{13} n \log r, \qquad \text{therefore} \qquad r < 8 \times 10^{13} n \log r.$$

Hence,

$$\begin{aligned} r &< 2 \times (8 \times 10^{13}) n \log(8 \times 10^{13} n) = 1.6 \times 10^{14} n (\log(8 \times 10^{13}) + \log n) \\ &< 1.6 \times 10^{14} n (32.1 + \log n) < 1.6 \times 10^{14} \times n(33 \log n) \\ &< 6 \times 10^{15} n \log n, \end{aligned} \tag{58}$$

where we used the fact that $n \geq 4$. With Lemma 3, we get that

$$\begin{aligned} \ell &\leq 7 \times 10^{10} n (n+r) \log(n+r) < 7 \times 10^{10} n (2r)(\log 2r) \\ &< 1.4 \times 10^{11} (nr)(\log 2 + \log r) \\ &< 1.4 \times 10^{11} \times (6 \times 10^{15})(n^2 \log n)(\log 2 + \log(6 \times 10^{15}) + 2 \log n) \\ &< 9 \times 10^{26} (n^2 \log n)(38 + 2 \log n) < 10^{27} \times (n^2 \log n)(40 \log n) \\ &< 4 \times 10^{28} n^2 (\log n)^2. \end{aligned} \tag{59}$$

Also,

$$\begin{aligned} k &< \ell(n+r+1) \leq \ell(2r) < 8 \times 10^{28} \times 6 \times 10^{15} (n^2 (\log n)^2)(n \log n) \\ &< 5 \times 10^{44} n^3 (\log n)^3. \end{aligned} \tag{60}$$

Now we assume that $\lambda_1 \notin \{(n+r)/2, (n+r)\ell/4\}$. Since $\ell < k$, we get that $\lambda_1 = \ell$. Hence, inequality (57) gives

$$\ell < 2 \times 10^{13} n \log r. \tag{61}$$

We next apply Matveev's theorem to get a lower bound on the expression appearing in the left–hand side of (49). We take $K := 4$, $\alpha_1 := F_{n-1}$, $\alpha_2 := \alpha$, $\alpha_3 := \sqrt{5}$, $\alpha_4 := \alpha^\ell - 1$. We also take $b_1 := -k$, $b_2 := (n+r+1)\ell$, $b_3 := -\ell$, $b_4 := -1$. Thus,

$$\Lambda_3 := F_{n-1}^{-k} \alpha^{(n+r+1)\ell} 5^{-\ell/2} (\alpha^\ell - 1)^{-1} - 1$$

is the expression appearing under the absolute value in the left–hand side of inequality (49). We have already checked that $\Lambda_3 \neq 0$. As for parameters, we have again $D := 2$ and we can take $A_1 := 2n \log \alpha$, $A_2 := 0.5$, $A_3 := 1.61$ and $B := 1.3 \times 10^{12} r^4$ as in the previous application of Matveev's theorem. As for A_4 observe that

$$\begin{aligned} Dh(\alpha_4) &= 2h(\alpha_4) \leq \log(\alpha^\ell - 1) + \max\{0, \log(|\beta^\ell - 1|)\} \\ &\leq \ell \log \alpha + \log 2 < (\log 2)(\ell + 1) < (2 \log 2)\ell < 1.4\ell, \end{aligned}$$

so we can take $A_4 := 1.4\ell$. Thus, the left–hand side of (49) is

$$|\Lambda_3| > \exp(-C_3(1+\log B)A_1A_2A_3A_4), \tag{62}$$

where

$$C_3 := 1.4 \times 30^7 \times 4^{4.5} \times 2^2 \times (1+\log 2) < 1.1 \times 10^{14}.$$

Thus, using part of the calculation from (55), we get that the expression under the exponential in (62) is bounded as

$$\begin{aligned} C_3(1+\log B)A_1A_2A_3A_4 &< 1.1 \times 10^{14}(1+\log(1.3\times 10^{12}r^4)) \times (2n\log\alpha) \\ &\times 0.5 \times 1.61 \times (1.4\ell) \\ &< 1.2 \times 10^{14}(n\ell)(29+4\log r) \\ &< 1.2 \times 10^{14}(n\ell)(8\log r) \\ &< 10^{15}(n\ell)\log r. \end{aligned} \tag{63}$$

Using also inequality (61), we get that

$$C_3(1+\log B)A_1A_2A_3A_4 < 10^{15} \times 2 \times 10^{13} n^2(\log r)^2 < 2 \times 10^{28} n^2 (\log r)^2. \tag{64}$$

We now compare bound (62) with bound (49) and use also inequality (53), to get that

$$\exp(-C_3(1+\log B)A_1A_2A_3A_4) < \frac{40}{1.5^{\lambda_2}},$$

where $\lambda_2 := \min\{k, (n+r)/2, (n+r)\ell/4\}$. Hence,

$$\lambda_2 < \frac{\log 40}{\log 1.5} + C_3(1+\log B)A_1A_2A_3A_4(\log 1.5)^{-1}. \tag{65}$$

We now distinguish the following cases.

Case 3. $\lambda_2 \in \{(n+r)/2, (n+r)\ell/4\}$.

In this case, we work with inequality (64) to get that inequality (65) implies that

$$r/4 \le \lambda_2 \le \frac{\log 40}{\log 1.5} + 2 \times 10^{28} \times (\log 1.5)^{-1} n^2 (\log r)^2 < 5 \times 10^{28} n^2 (\log r)^2,$$

giving

$$r < 2 \times 10^{29} n^2 (\log r)^2.$$

One checks easily that if $A > 100$, then the inequality

$$\frac{x}{(\log x)^2} < A$$

implies $x < 4A(\log A)^2$. Indeed, for if not, since the function $x/(\log x)^2$ is increasing for $x > e^2$, it follows that

$$\frac{4A(\log A)^2}{(\log(4A(\log A)^2))^2} \leq \frac{x}{(\log x)^2} < A,$$

giving $2\log A < \log(4A(\log A)^2)$, or $A^2 < 4A\log(A)^2$, or $A < 4(\log A)^2$, which gives $A < 75$. Applying this with $A := 2 \times 10^{29}n^2$, we get that

$$\begin{aligned} r &< 4 \times (2 \times 10^{29})n^2(\log(2 \times 10^{29}n^2))^2 \\ &< 8 \times 10^{29}n^2(\log(2 \times 10^{29}) + 2\log n)^2 \\ &< 8 \times 10^{29}n^2(68 + 2\log n)^2 < 8 \times 10^{29}n^2(70\log n)^2 \\ &< 4 \times 10^{33}n^2(\log n)^2. \end{aligned} \tag{66}$$

With inequality (61), we get

$$\begin{aligned} \ell &< 2 \times 10^{13}n\log(4 \times 10^{33}n^4) < 2 \times 10^{13}(78 + 4\log n) \\ &< 2 \times 10^{13}n(82\log n) < 2 \times 10^{15}n\log n. \end{aligned} \tag{67}$$

Finally, using inequalities (66), (67) and (21), we also get

$$k \leq \ell(n + r + 1) \leq 2\ell nr \leq 2 \times 10^{15} \times 4 \times 10^{33}n^3(\log n)^3 = 8 \times 10^{48}n^3(\log n)^3. \tag{68}$$

Now we move on to the last case, namely:

Case 4. $\lambda_2 = k$.

Then inequality (65) together with inequality (63) gives

$$k < \frac{\log 40}{\log 1.5} + 10^{15} \times (\log 1.5)^{-1}n\ell\log r < 2.5 \times 10^{15}n\ell\log r.$$

From Lemma 2, we deduce that

$$\begin{aligned} \ell r &< \ell(n + r) < k(n + 2) < 2.5 \times 10^{15}n(n + 2)\ell\log r \\ &\leq 2.5 \times 10^{15}n(1.5n)\ell\log r < 4 \times 10^{15}n^2\ell\log r, \end{aligned}$$

giving

$$r < 4 \times 10^{15}n^2\log r.$$

Hence,

$$\begin{aligned} r &< 2 \times 4 \times 10^{15}n^2\log(4 \times 10^{15}n^2) < 8 \times 10^{15}n^2(36 + 2\log n) \\ &< 8 \times 10^{15}(26\log n) < 2.1 \times 10^{17}n^2\log n. \end{aligned} \tag{69}$$

With inequality (61), we get that

$$\begin{aligned} \ell &< 2 \times 10^{13}n\log(2.1 \times 10^{17}n^3) < 2 \times 10^{13}n(40 + 3\log n) \\ &< 2 \times 10^{13}n(32\log n) < 7 \times 10^{14}n\log n, \end{aligned} \tag{70}$$

while by (21), (69) and (70), we get

$$\begin{aligned} k &\leq \ell(n+r+1) \leq 2\ell r < 2 \times 7 \times 10^{14} \times 2.1 \times 10^{17}(n \log n)(n^2 \log n) \\ &< 3 \times 10^{32} n^3 (\log n)^2. \end{aligned} \tag{71}$$

Let us summarize what we have done. The following lemma follows by picking up the worse upper bounds for r, ℓ and k from estimates (52), (58), (66), (69) (for r), (50), (59), (67), (70) (for ℓ) and (51), (60), (68) and (71) (for k), respectively.

Lemma 6. *If (k, ℓ, n, r) is a solution of equation (2) with $n + r \geq 3002$, then the estimates*

$$r \leq 10^{34} n^2 (\log n)^2; \tag{72}$$

$$\ell \leq 10^{29} n^2 (\log n)^2; \tag{73}$$

$$k \leq 10^{49} n^3 (\log n)^3 \tag{74}$$

hold.

6. The case of the small n

Here, we treat the case when $n \leq 3000$. Then, by Lemmas 2 and 6, we have

$$(n+r)\ell \leq k(n+2) \leq 10^{49} n^3 (\log n)^3 (n+2) < 5 \times 10^{65} \qquad \text{for} \quad n \leq 3000. \tag{75}$$

From inequality (48), we infer that

$$\left| F_{n-1}^{-k} \alpha^{(n+r)\ell} 5^{-\ell/2} - 1 \right| < \frac{60}{1.5^{\lambda_3}},$$

where

$$\lambda_3 := \min\{\ell, k, (n+r)/2, (n+r-t)\ell/2\}. \tag{76}$$

Put

$$\begin{aligned} \Lambda_4 &:= F_{n-1}^{-k} \alpha^{(n+r)\ell} 5^{-\ell/2} - 1, \\ \Gamma_4 &:= -k \log F_{n-1} - (n+r)\ell \log \alpha - \ell \log \sqrt{5}. \end{aligned} \tag{77}$$

Assume that $\lambda_3 \geq 13$. We then have that

$$|\mathrm{e}^{\Gamma_4} - 1| = |\Lambda_4| < \frac{60}{1.5^{\lambda_3}} < \frac{1}{3},$$

which gives that $e^{|\Gamma_4|} < 3/2$. Hence,

$$|\Gamma_4| < \mathrm{e}^{|\Gamma_4|} |\mathrm{e}^{\Gamma_4} - 1| < 1.5 |\Lambda_4| < \frac{100}{1.5^{\lambda_3}}.$$

Observe that Γ_4 is an expression of the form

$$|x \log F_{n-1} + y \log \alpha + z \log \sqrt{5}|, \tag{78}$$

where $x := -k,\ y := (n+r)\ell,\ z := -\ell$ are integers with $\max\{|x|, |y|, |z|\} \leq 5 \times 10^{65}$ (see (75)). For each $n \in [4, 3000]$, we used the LLL algorithm to compute a lower bound for the smallest nonzero number of the form (78) with integer coefficients $x,\ y,\ z$ not exceeding 5×10^{65} in absolute value. We followed the method described in [5, Section 2.3.5], which provides such bound using the approximation for the shortest vector in the corresponding lattice obtained by LLL algorithm. In these computations, we used the PARI/GP function `qflll`. The minimal such value is $> 100/1.5^{750}$, which gives that $\lambda_3 \leq 750$. Observe that since $n \leq 3000$ and we already covered the range when both n and r were in $[1, 3000]$, it follows that $r > 3000$. In particular, $(n+r)/2 > 1500$ and $r > n$, therefore $t = \lfloor (n+r)/2 \rfloor$. Thus, $(n+r-t)\ell/2 \geq (n+r)\ell/4 > \ell$. Since also $k > \ell$, we learn from this computation that $\ell = \lambda_3$ and $\ell \leq 750$.

Next we move to inequality (49) and rewrite it as

$$\left|F_{n-1}^{-k}\alpha^{(n+r)\ell}5^{-\ell/2}(\alpha^\ell - 1) - 1\right| < \frac{40}{1.5^{\lambda_4}},$$

where

$$\lambda_4 := \min\{k, (n+r)/2, (n+r-t)\ell/2\}. \tag{79}$$

We now put

$$\begin{aligned} \Lambda_5 &:= F_{n-1}^{-k}\alpha^{(n+r)\ell}5^{-\ell/2}(\alpha^\ell - 1) - 1, \\ \Gamma_5 &:= -k \log F_{n-1} + (n+r)\log \alpha^\ell + \log(5^{-\ell/2}(\alpha^\ell - 1). \end{aligned} \tag{80}$$

Assume that $\lambda_4 \geq 12$. We then have that

$$|\mathrm{e}^{\Gamma_5} - 1| = |\Lambda_5| < \frac{40}{1.5^{\lambda_4}} < \frac{1}{3},$$

which gives that $e^{|\Gamma_5|} < 3/2$. Hence,

$$|\Gamma_5| < \mathrm{e}^{|\Gamma_5|}|\mathrm{e}^{\Gamma_5} - 1| < 1.5|\Lambda_5| < \frac{60}{1.5^{\lambda_4}}.$$

Observe that $|\Gamma_5|$ is an expression of the form

$$|x \log \alpha_1 + y \log \alpha_2 + \log \alpha_3|, \tag{81}$$

where $\alpha_1 := F_{n-1},\ \alpha_2 := \alpha^\ell,\ \alpha_3 = 5^{-\ell/2}(\alpha^\ell - 1)$, and $x := -k,\ y := (n+r)$. Since $n \leq 3000$, by Lemma 6, we have that

$$\max\{|x|, |y|\} \leq 10^{49}(n \log n)^3 < 2 \times 10^{62}. \tag{82}$$

For each $n \in [4, 3000]$ and each $\ell \in [1, 750]$, we performed the LLL algorithm to find a lower bound on the smallest number of the form (81) whose coefficients $x,\ y$ are integers satisfying

(82). In each case, we got that this lower bound is $> 60/1.5^{-800}$, which gives that $\lambda_4 \leq 800$. Again, we have $(n+r)/2 > 1500$; hence, $\lambda_4 = k$ or $\lambda_4 = (n+r-t)\ell/2$. From what we have seen, $(n+r-t)\ell/2 \geq (n+r)\ell/4 > 750\ell$ and this last number is $\geq 1500 > 800$ unless $\ell = 1$. Thus, we always have $k \leq 800$, unless $\ell = 1$, and then $(n+r)/4 \leq 800$, which gives $n + r \leq 3200$.

We first deal with the second possibility. Fix $n \leq 3000$ and r such that $n + r \leq 4000$. Let $\ell = 1$. Then

$$N = F_{n+1} + \cdots + F_{n+r}$$

is known. Furthermore, by inequality (11) and estimate (12), we get that $N = F_{n-1}^k(1+S)$, where $0 < S < 3/1.5^k < 2/3$ for $k \geq 4$. Thus, $(3/5)N < F_{n-1}^k < N$, therefore

$$\frac{\log N - \log(5/3)}{\log F_{n-1}} < k < \frac{\log N}{\log F_{n-1}}. \tag{83}$$

For n and r fixed, there is at most one k satisfying inequalities (83). When this exists, we tested whether with this value of k the quadruple $(k, 1, n, r)$ does indeed satisfy equation (2). No new solution turned up.

So, from now on, we assume that $k \leq 800$. Thus,

$$\ell(n+r) \leq k(n+2) \leq 800 \times 3002 < 2.5 \times 10^6.$$

We now fix again n and apply again the LLL algorithm to get a lower bound on the minimum absolute value of the nonzero numbers of the form (78) where now x, y, z are integer coefficients of absolute values $\leq 2.5 \times 10^6$. In all cases, we got a lower bound of $100/1.5^{100}$, which gives that $\lambda_3 \leq 100$. Hence, $\ell \leq 100$. We now moved on to the number of the form (81), and for all values of $n \in [4, 3000]$ and each $\ell \in [1, 100]$, we applied the LLL algorithm to find a lower bound for the absolute value of the nonzero numbers of the form (81) when x, y are integer coefficients not exceeding 2.5×10^6 in absolute values. In all cases, we got a lower bound larger than $60/1.5^{130}$, showing that $k \leq 130$.

Now we covered the rest by brute force. That is, suppose that $n \in [4, 3000]$ and $\ell < k$ are fixed in $[1, 100]$ and $[4, 130]$, respectively. Then $N = F_1^k + \cdots + F_{n-1}^k$ is fixed and $(3/5)N < F_{n+r}^{\ell} < N$. Thus,

$$(3N/5)^{1/\ell} < F_{n+r} < N^{1/\ell}. \tag{84}$$

For each fixed triple (k, ℓ, n), the above inequality gives some range for r. For each of these candidates r such that $r \geq 3001$, we checked whether the quadruple (k, ℓ, n, r) does indeed satisfy (2). As expected, no new solutions turned up.

This completes the analysis of the case when $n \leq 3000$. We record our conclusion as follows.

Lemma 7. *If (k, ℓ, n, r) is a solution of (2) other than $(8, 2, 4, 3)$, then $n \geq 3001$. Furthermore, if $\ell = 1$, then $n + r \geq 4001$.*

7. Three linear forms in logarithms

Now we start working on the left–hand side of equation (2) and do to it what we did to the right–hand side of it in Section 3. Write $m := \lfloor n/2 \rfloor$, and put

$$N_4 := \sum_{j \le m} F_j^k.$$

Then

$$N_4 < \left(\sum_{j \le m} F_j\right)^k < (F_{m+2})^k < \left(\frac{F_{n-1}}{F_{n-m-2}}\right)^k < \frac{N}{\alpha^{(n-m-4)k}} \le \frac{N}{\alpha^{(n/2-4)k}} < \frac{N}{\alpha^n}, \tag{85}$$

because $n \ge 3001$ and $k \ge 4$.

Assume now that $j \in [m+1, n-1]$. Formula (29) gives us that

$$F_j^k = \left(\frac{\alpha^j - \beta^j}{5^{1/2}}\right)^k = \frac{\alpha^{jk}}{5^{k/2}} \left(1 - \frac{(-1)^j}{\alpha^{2j}}\right)^k.$$

By Lemma 6, we have that

$$\frac{k}{\alpha^{2j}} \le \frac{10^{49} n^3 (\log n)^3}{\alpha^n} < \frac{1}{\alpha^{n/2}}.$$

The last inequality holds whenever $n \ge 572$, which is the case for us. Set $y := 1/\alpha^{n/2}$. Since $n \ge 3001$, it follows that $y < \alpha^{-1500} < 10^{-300}$. The argument used to prove inequality (32), based on the inequalities (30) and (31) yields that

$$\left|\left(1 - \frac{(-1)^j}{\alpha^{2j}}\right)^k - 1\right| < \frac{2k}{\alpha^{2j}} < 2y,$$

therefore

$$\left|F_j^k - \frac{\alpha^{jk}}{5^{k/2}}\right| = \frac{\alpha^{jk}}{5^{k/2}} \left|\left(1 - \frac{(-1)^j}{\alpha^{2j}}\right)^k - 1\right| < 2y \left(\frac{\alpha^{jk}}{5^{k/2}}\right).$$

Since y is small, we get, as in (33), that the inequality above implies that $\alpha^{jk}/5^{k/2} < 1.5F_j^k$, therefore the inequality

$$\left|F_j^k - \frac{\alpha^{jk}}{5^{k/2}}\right| < 3yF_j^k \tag{86}$$

holds for all $j \in [m+1, n-1]$. Now we sum up the above inequalities over all j getting that if we put

$$N_5 := \sum_{j=m+1}^{n-1} \frac{\alpha^{jk}}{5^{k/2}},$$

then the inequality

$$\begin{aligned} |N - N_4 - N_5| &= \left| \sum_{j=m+1}^{n-1} F_j^k - \sum_{j=m+1}^{n-1} \frac{\alpha^{kj}}{5^{k/2}} \right| \le \sum_{j=m+1}^{n-1} \left| F_j^k - \frac{\alpha^{jk}}{5^{k/2}} \right| \\ &< 3y \sum_{j=m+1}^{n-1} F_j^k < 3yN \end{aligned} \tag{87}$$

holds. We now estimate N_5. Clearly,

$$N_5 = \frac{\alpha^{nk} - \alpha^{(m+1)k}}{5^{k/2}(\alpha^k - 1)}.$$

Note that

$$\begin{aligned} \frac{\alpha^{(m+1)k}}{5^{k/2}(\alpha^k - 1)} &\le \frac{1.5F_{m+1}^k}{\alpha^4 - 1} < F_{m+1}^k < \left(\frac{F_{n-1}}{F_{n-m-1}} \right)^k \\ &< \frac{N}{\alpha^{(n-m-3)k}} \le \frac{N}{\alpha^{(n-6)k/2}} < \frac{N}{\alpha^n}. \end{aligned} \tag{88}$$

From inequalities (85), (87) and (88), we get

$$\begin{aligned} \left| N - \frac{\alpha^{nk}}{5^{k/2}(\alpha^k - 1)} \right| &\le N_4 + |N - N_4 - N_5| + \frac{\alpha^{(m+1)k}}{5^{k/2}(\alpha^k - 1)} \\ &< N \left(\frac{1}{\alpha^n} + \frac{3}{\alpha^{n/2}} + \frac{1}{\alpha^n} \right) < \frac{4N}{\alpha^{n/2}}. \end{aligned} \tag{89}$$

Multiplying both sides above by $\alpha^k - 1$, we also get that

$$\left| N(\alpha^k - 1) - \frac{\alpha^{nk}}{5^{k/2}} \right| < \frac{4(\alpha^k - 1)N}{\alpha^{n/2}} < \frac{4N}{\alpha^{n/2-k}},$$

therefore

$$\left| N\alpha^k - \frac{\alpha^{nk}}{5^{k/2}} \right| < N \left(1 + \frac{4}{\alpha^{n/2-k}} \right).$$

Hence,

$$\left| N - \frac{\alpha^{(n-1)k}}{5^{k/2}} \right| < N \left(\frac{1}{\alpha^k} + \frac{4}{\alpha^{n/2}} \right) < 4 \left(\frac{1}{\alpha^k} + \frac{1}{\alpha^{n/2}} \right). \tag{90}$$

Let

$$A := \frac{\alpha^{nk}}{5^{k/2}(\alpha^k - 1)} \qquad \text{and} \qquad B := \frac{\alpha^{(n-1)k}}{5^{k/2}}.$$

From inequalities (89) and (90) together with the fact that $k \ge 4$ and $n \ge 3001$, which implies that

$$4 \left(\frac{1}{\alpha^k} + \frac{1}{\alpha^{n/2}} \right) < 0.6,$$

we infer that both inequalities $N < 2.5A$ and $N < 2.5B$ hold. Now we put together the two inequalities (89) and (90) involving N together with the three inequalities (42), (43) and (44) involving also N, and get the following six inequalities:

$$\begin{aligned}
\left|\frac{\alpha^{(n-1)k}}{5^{k/2}} - \frac{\alpha^{(n+r)\ell}}{5^{\ell/2}}\right| &< 8N\left(\frac{1}{\alpha^k} + \frac{1}{\alpha^{n/2}} + \frac{1}{\alpha^\ell} + \frac{1}{\alpha^{(n+r)/2}} + \frac{1}{\alpha^{(n+r-t)\ell/2}}\right)\\
&< 16N\left(\frac{1}{\alpha^\ell} + \frac{1}{\alpha^{n/2}} + \frac{1}{\alpha^{(n+r-t)\ell/2}}\right).\\
\left|\frac{\alpha^{(n-1)k}}{5^{k/2}} - \frac{\alpha^{(n+r+1)\ell}}{5^{\ell/2}(\alpha^\ell-1)}\right| &< 8N\left(\frac{1}{\alpha^k} + \frac{1}{\alpha^{n/2}} + \frac{1}{\alpha^{(n+r)/2}} + \frac{1}{\alpha^{(n+r-t)\ell/2}}\right)\\
&< 16N\left(\frac{1}{\alpha^k} + \frac{1}{\alpha^{n/2}} + \frac{1}{\alpha^{(n+r-t)\ell/2}}\right).\\
\left|\frac{\alpha^{(n-1)k}}{5^{k/2}} - \frac{\alpha^{(t+1)\ell}(\alpha^{(n+r-t)\ell}-1)}{5^{\ell/2}(\alpha^\ell-1)}\right| &< 8N\left(\frac{1}{\alpha^k} + \frac{1}{\alpha^{n/2}} + \frac{1}{\alpha^{(n+r)/2}}\right)\\
&< 16N\left(\frac{1}{\alpha^k} + \frac{1}{\alpha^{n/2}}\right);\\
\left|\frac{\alpha^{nk}}{5^{k/2}(\alpha^k-1)} - \frac{\alpha^{(n+r)\ell}}{5^{\ell/2}}\right| &< 8N\left(\frac{1}{\alpha^{n/2}} + \frac{1}{\alpha^\ell} + \frac{1}{\alpha^{(n+r)/2}} + \frac{1}{\alpha^{(n+r-t)\ell/2}}\right)\\
&< 16N\left(\frac{1}{\alpha^\ell} + \frac{1}{\alpha^{n/2}} + \frac{1}{\alpha^{(n+r-t)\ell/2}}\right).\\
\left|\frac{\alpha^{nk}}{5^{k/2}(\alpha^k-1)} - \frac{\alpha^{(n+r+1)\ell}}{5^{\ell/2}(\alpha^\ell-1)}\right| &< 8N\left(\frac{1}{\alpha^{n/2}} + \frac{1}{\alpha^{(n+r)/2}} + \frac{1}{\alpha^{(n+r-t)\ell/2}}\right)\\
&< 16N\left(\frac{1}{\alpha^{n/2}} + \frac{1}{\alpha^{(n+r-t)\ell/2}}\right).\\
\left|\frac{\alpha^{nk}}{5^{k/2}(\alpha^k-1)} - \frac{\alpha^{(t+1)\ell}(\alpha^{(n+r-t)\ell}-1)}{5^{\ell/2}(\alpha^\ell-1)}\right| &< 8N\left(\frac{1}{\alpha^{n/2}} + \frac{1}{\alpha^{(n+r)/2}}\right)\\
&< \frac{16N}{\alpha^{n/2}}.
\end{aligned}$$

We will actually not use the third, fourth or sixth inequality above, we will only use the first, second and fifth. Since each one of them involves either A or B (but not both), it follows that dividing both sides of the respective inequality by its A or B term and using the fact that $N < 2.5\max\{A, B\}$, we get the following three inequalities

$$\left|\alpha^{(n+r)\ell-(n-1)k}5^{(k-\ell)/2} - 1\right| < 40\left(\frac{1}{\alpha^\ell} + \frac{1}{\alpha^{n/2}} + \frac{1}{\alpha^{(n+r-t)\ell/2}}\right); \tag{91}$$

$$\left|\alpha^{(n+r+1)\ell-(n-1)k}5^{(k-\ell)/2}(\alpha^\ell-1)^{-1} - 1\right| < 40\left(\frac{1}{\alpha^k} + \frac{1}{\alpha^{n/2}} + \frac{1}{\alpha^{(n+r-t)\ell/2}}\right); \tag{92}$$

$$\left|\alpha^{(n+r+1)\ell-nk}5^{(k-\ell)/2}\left(\frac{\alpha^k-1}{\alpha^\ell-1}\right) - 1\right| < 40\left(\frac{1}{\alpha^{n/2}} + \frac{1}{\alpha^{(n+r-t)\ell/2}}\right). \tag{93}$$

We next comment on the size of $(n+r-t)\ell$. Assume first that $r \ge n$. Then $t = \lfloor (n+r)/2,$

therefore
$$(n+r-t)\ell \geq (n+r-\lfloor (n+r)/2 \rfloor)\ell \geq (n+r)\ell/2 \geq n/2.$$
Otherwise, we have $t = n$, therefore $(n+r-t)\ell = r\ell$. But note that Lemma 2 tells us that the inequality
$$k(n-1) \leq \ell(n+r) + 2k + \ell$$
holds. This can be rewritten as
$$k(n-3) \leq \ell n + r\ell + \ell = \ell(n-3) + r\ell + 4\ell \leq \ell(n-3) + 5r\ell,$$
giving that
$$(n-3)(k-\ell) \leq 5r\ell.$$
Hence, $(n+r-t)\ell = r\ell \geq (n-3)(k-\ell)/5$ holds when $r < n$. To summarize, we always have
$$(n+r-t)\ell/2 \geq (n-3)(k-\ell)/10. \tag{94}$$
In conclusion, on the right hand–side of inequalities (91)–(93), the exponent of α in last term which is $(n+r-t)\ell/2$ is always of comparable size, at least, with the exponent of α in the previous term which is $n/2$. We record the conclusions of this section as follows.

Lemma 8. *If (k, ℓ, n, r) is a positive integer solution of equation (2) other than $(8, 2, 4, 3)$, then inequalities (91)–(93) hold. Moreover, $(n+r-t)\ell/2 \geq (n-3)(k-\ell)/10$.*

8. Logarithmic bounds for ℓ and k

Our next goal is to bound ℓ and k as logarithmic functions in n. This will be achieved by applying Matveev's theorem to bound from below the left–hand sides of (91)–(93). Let us see whether there are instances in which the left–hand side of one of these three inequalities can be zero.

If the left–hand side of (91) is zero, we then get that $\alpha^{2(n-1)k-2(n+r)\ell} = 5^{k-\ell}$. Since no power of α of nonzero integer exponent can be an integer, it follows that $(n-1)k = (n+r)\ell$ and $k = \ell$, but this is impossible.

If the left–hand side of (92) is zero, we then get that
$$\alpha^{(n+r+1)\ell-(n-1)k} 5^{(k-\ell)/2} = \alpha^{\ell} - 1. \tag{95}$$
Conjugating the above relation in $\mathbb{Q}(\sqrt{5})$ and multiplying the two resulting equations we get
$$(-1)^{(n+r+1)\ell-(n-1)k+(k-\ell)} 5^{k-\ell} = (\alpha^{\ell}-1)(\beta^{\ell}-1) = -\alpha^{\ell} - \beta^{\ell} + 1 + (-1)^{\ell}. \tag{96}$$
It is well–known that $\alpha^m + \beta^m = L_m$, where $(L_m)_{m\geq 0}$ is the Lucas sequence given by $L_0 = 0$, $L_1 = 1$ and $L_{m+2} = L_{m+1} + L_m$ for all $m \geq 0$. Hence, equation (96) above is

$L_\ell - 1 - (-1)^\ell = \pm 5^{k-\ell}$. If ℓ is odd, we then get $L_\ell = \pm 5^{k-\ell}$, which is impossible since no member of the Lucas sequence is a multiple of 5. If ℓ is even, then $L_\ell \geq L_2 = 3$, so that $L_\ell - 1 - (-1)^\ell = L_\ell - 2$ is positive. If $\ell/2$ is odd, then

$$5^{k-\ell} = L_\ell - 2 = \alpha^\ell + \beta^\ell - 2 = \alpha^\ell + \beta^\ell + 2(\alpha\beta)^{\ell/2} = (\alpha^{\ell/2} + \beta^{\ell/2})^2 = L_{\ell/2}^2,$$

which is also impossible since $L_{\ell/2}$ cannot be a multiple of 5. Finally, if ℓ is a multiple of 4, we then get that

$$5^{k-\ell} = L_\ell - 2 = (\alpha^{\ell/2} - \beta^{\ell/2})^2 = 5F_{\ell/2}^2,$$

so $F_{\ell/2}^2 = 5^{k-\ell-1}$. By the Primitive Divisor Theorem, the only Fibonacci numbers which are powers of 5 are $1 = F_1 = F_2$ and $5 = F_5$. Thus, $\ell/2 = 2$, therefore $\ell = 4$ and $k - \ell = 1$, so $k = 5$. Since $\alpha^4 - 1 = \sqrt{5}\alpha^2$, equation (95) also gives $4(n + r + 1) - 5(n - 1) = 2$, therefore $n = 4r + 7$.

The conclusion is that the left–hand side of inequality (92) is nonzero except when $(k, \ell, n, r) = (5, 4, 4r + 7, r)$. However, later we shall use inequality (91) to get some bound for ℓ, and then inequality (92) to get some bound for k. If ℓ have already been bounded (like it is the case when $\ell = 4$ and $k = 5$), then we will move on to inequality (93).

Let us now check that the left–hand side of inequality (93) is nonzero. Assuming that it is, we get the equation

$$\alpha^{(n+r+1)\ell - nk} \left(\frac{\alpha^k - 1}{\alpha^\ell - 1}\right) = \frac{1}{5^{(k-\ell)/2}}.$$

Conjugating the above relation in $\mathbb{Q}(\sqrt{5})$ and multiplying the two resulting relations, we get

$$\left|\frac{L_k - 1 - (-1)^k}{L_\ell - 1 - (-1)^\ell}\right| = \left|\frac{(\alpha^k - 1)(\beta^k - 1)}{(\alpha^\ell - 1)(\beta^\ell - 1)}\right| = \frac{1}{5^{k-\ell}} \leq \frac{1}{5}.$$

Hence, since $k \geq 4$, so $L_k \geq 7$, we get that

$$5L_k \leq 5|L_k - 1 - (-1)^k| \leq |L_\ell - 1 - (-1)^\ell| \leq L_\ell + 2 < L_k + 2,$$

giving $4L_k < 2$, which is impossible. Hence, the left–hand side of inequality (93) cannot be zero.

We now apply Matveev's theorem to the left–hand sides of inequalities (91)–(93).

We start with bounding ℓ by applying Matveev's theorem to inequality (91). We already checked that this is nonzero. Let $\lambda_5 := \min\{\ell, n/2, (n + r - t)\ell/2\}$, and note that

$$(n + r - t)\ell/2 \geq \min\{n/2, (n - 3)(k - \ell)/10\}$$

(see Lemma 8). From inequality (91), we obtain

$$\left|\alpha^{(n+r)\ell-(n-1)k} 5^{(k-\ell)/2} - 1\right| < \frac{120}{\alpha^{\lambda_5}}. \tag{97}$$

We apply Matveev's theorem to the left–hand side of the above inequality with $K := 2$, $\alpha_1 := \alpha$, $\alpha_2 := \sqrt{5}$, $b_1 := (n+r)\ell - (n-1)k$, $b_2 := k - l$, $D := 2$. We take as in prior applications of this theorem $A_1 := \log\alpha$, $A_2 := \log 5$. Furthermore, by (6), we have that $\max\{|b_1|, |b_2|\} \le 2k + \ell$, and by (73) and (74), we may take $B := 2.01 \times 10^{49} n^3 (\log n)^3$. Now, Matveev's theorem and inequality (97) give us that

$$\lambda_5 < \frac{\log 120}{\log\alpha} + 8.4 \times 10^9 (1 + \log(2.01 \times 10^{49} n^3 (\log n)^3).$$

Since we know that $n \ge 3001$, we get that

$$\lambda_5 < 1.59 \times 10^{11} \log n.$$

Assume that $\lambda_5 \ne \ell$. Then $\lambda_5 \ge (n-3)(k-\ell)/10$. But then

$$n < 3 + 1.59 \times 10^{12} \log n,$$

which gives that $n < 5.02 \times 10^{13}$. Now we apply continued fractions to a variant of the inequality (97), which is

$$\left| \frac{\log\sqrt{5}}{\log\alpha} - \frac{(n-1)k - (n+r)\ell}{k-\ell} \right| < \frac{240}{\alpha^{\lambda_5}(k-\ell)\log\alpha} < \frac{1}{2(k-\ell)^2}. \tag{98}$$

The above inequality holds because $\lambda_5 \ge (n-3)(k-\ell)/10 \ge 299(k-\ell)$. Hence, we get that $((n-1)k - (n+r)\ell)/(k-\ell) = p_i/q_i$ for some convergent p_i/q_i of $\gamma := \log\sqrt{5}/\log\alpha$. We computed all convergents p_i/q_i of γ satisfying $q_i < 4 \times 10^{94}$, which is an upper bound for $k - \ell$ when $n < 5.02 \times 10^{13}$ by Lemma 6. We find that all convergents in that range satisfy

$$|q_i \log\sqrt{5} - p_i \log\alpha| > \frac{240}{\alpha^{470}}.$$

Hence, we conclude that $\lambda_5 \le 469$. Since $\lambda_5 \ge (n-3)/2$, we get that $n \le 938$, which contradicts previously established result that $n \ge 3001$. Thus, we have shown that $\lambda_5 = \ell$, and so

$$\ell < 1.59 \times 10^{11} \log n. \tag{99}$$

We will now establish a similar logarithmic bound for k using inequality (92). The cases $\ell = 1$ and $\ell = 2$ will be treated separately because in these cases $\alpha^\ell - 1$ is a power of α. From (92), for $\ell = 1$ we get

$$\left|\alpha^{n+r+2-(n-1)k} 5^{(k-\ell)/2} - 1\right| < \frac{120}{\alpha^{\lambda_6}},$$

while for $\ell = 2$ we get

$$\left|\alpha^{2n+2r+1-(n-1)k} 5^{(k-\ell)/2} - 1\right| < \frac{120}{\alpha^{\lambda_6}},$$

where $\lambda_6 := \min\{k, n/2, (n+r-t)\ell/2\}$. The left–hand sides of the two inequalities above are nonzero by the argument used to prove that the left–hand side of inequality (91) is nonzero, since assuming that it were zero, we would get that $k = \ell$, which is not allowed. Thus, we may apply again Matveev's theorem as previously with $\alpha_1 := \alpha$, $\alpha_2 := \sqrt{5}$. After some calculation, we get the same bound for λ_6 as the bound obtained previously for λ_5. Hence, $\lambda_6 < 1.59 \times 10^{11} \log n$ in this case. The assumption that $\lambda_6 \neq k$ leads to a contradiction as before, and thus we obtain that for $\ell \in \{1, 2\}$ we have

$$k < 1.59 \times 10^{11} \log n. \tag{100}$$

Assume now that $\ell \geq 3$ but $(\ell, k) \neq (4, 5)$. From inequality (92), we get

$$\left|\alpha^{(n+r+1)\ell-(n-1)k} 5^{(k-\ell)/2} (\alpha^{\ell} - 1)^{-1} - 1\right| < \frac{120}{\alpha^{\lambda_6}}. \tag{101}$$

We apply Matveev's theorem to inequality (101). We have $K := 3$, $\alpha_1 := \alpha$, $\alpha_2 := \sqrt{5}$, $\alpha_3 = (\alpha^{\ell} - 1)$, $b_1 := (n + r + 1)\ell - (n - 1)k$, $b_2 := k - l$, $b_3 := -1$, $D := 2$. We take $A_1 := \log \alpha$, $A_2 := \log 5$, $A_3 := \ell$. Here we use that $\ell \geq 3$. Note that, by (99), we have $A_3 < 1.59 \times 10^{11} \log n$. Furthermore, by inequality (6), we have $\max\{|b_1|, |b_2|, |b_3|\} \leq 2k + 2\ell$, and by inequalities (73) and (73) we may take $B := 2.01 \times 10^{49} n^3 (\log n)^3$. Now, Matveev's theorem and inequality (101) give us that

$$\lambda_6 < \frac{\log 120}{\log \alpha} + 2.49 \times 10^{23} (1 + \log(2.01 \times 10^{49} n^3 (\log n)^3) \log n.$$

Since we know that $n \geq 3001$, we get that

$$\lambda_6 < 3.58 \times 10^{25} (\log n)^2. \tag{102}$$

Assume that $\lambda_6 \neq k$. Then $\lambda_6 \geq (n - 3)/10$. But then

$$n < 3 + 3.58 \times 10^{26} (\log n)^2,$$

which gives that $n < 1.74 \times 10^{30}$. The application of the above described continued fraction method for inequality (98) in this new range for n gives that $\ell = \lambda_5 \leq 705$.

Now we apply the LLL algorithm, as explained in Section 6, to find a lower bound for the smallest nonzero value of a number of form

$$|x \log \alpha + y \log \sqrt{5} \pm \log(\alpha^{\ell} - 1)|, \tag{103}$$

with $\max\{|x|, |y|\} < 5.14 \times 10^{143}$, which is the bound for

$$|(n + r + 1)\ell - (n - 1)k| \leq 2k + 2\ell \qquad \text{when} \qquad n < 1.74 \times 10^{30},$$

by Lemma 6. The computation shows that this minimal value is $> 240/\alpha^{1400}$, which gives that $\lambda_6 \leq 1400$. If $\lambda_6 = n/2$, we get $n \leq 2880$, contradicting the bound $n \geq 3000$. If $\lambda_6 = (n + r - t)\ell/2$, then from $(n - 3)(k - \ell)/10 \leq \lambda_6 \leq 1400$ and $n \geq 3000$, we get that $k - \ell \leq 4$ and

$$k < 1.6 \times 10^{11} \log n. \tag{104}$$

It remains to treat the case when $\lambda_6 = k$. Then, by (102), we have

$$k < 3.58 \times 10^{25} (\log n)^2. \tag{105}$$

We summarize the results of this section in the following lemma.

Lemma 9. *Let* (k, ℓ, n, r) *be a solution of equation* (2). *Then*

$$\ell < 1.59 \times 10^{11} \log n;$$
$$k < 3.58 \times 10^{25} (\log n)^2.$$

If $\ell \in \{1, 2\}$, *then*

$$k < 1.59 \times 10^{11} \log n.$$

9. Absolute upper bound for n and the end of the proof

Now that we have upper bounds for ℓ and k as logarithmic functions of n (see Lemma 9), we may apply Matveev's theorem to the left–hand side of inequality (93), in order to obtain an absolute upper bound for n. We have already checked that this expression is nonzero. We take $K := 3$, $\alpha_1 := \alpha$, $\alpha_2 := \sqrt{5}$, $\alpha_3 := (\alpha^k - 1)/(\alpha^\ell - 1)$, $b_1 := (n + r + 1)\ell - nk$, $b_2 := k - l$, $b_3 := 1$, $D := 2$. We also take $A_1 := \log \alpha$, $A_2 := \log 5$, $A_3 := k$. By Lemma 9, we have $A_3 < 3.58 \times 10^{25} (\log n)^2$ and by (6), $\max\{|b_1|, |b_2|, |b_3|\} \leq 3k + 2\ell$, so by (73) and (73) we may take $B := 1.08 \times 10^{26} (\log n)^2$. Now, Matveev's theorem and inequality (93) give us that

$$\min\{n/2, (n + r - t)\ell/2\} <$$
$$(\log 80) \log \alpha + 5.59 \times 10^{37} (1 + \log(1.08 \times 10^{26} (\log n)^2))(\log n)^2.$$

Since, $(n + r - t)\ell/2 \geq (n - 3)/10$, we get an absolute upper bound for n, namely

$$n < 4.15 \times 10^{44}. \tag{106}$$

Inserting the bound for n given by (106) in the bound for k from Lemma 9, we get

$$k < 3.78 \times 10^{29}.$$

Applying the continued fraction method to inequality (98) for

$$k - \ell < 3.78 \times 10^{29}$$

gives that $\ell = \lambda_5 \leq 155$. Applying the LLL algorithm to the numbers of the form (103) with the bounds $k < 3.78 \times 10^{29}$ and $\ell < 155$ gives that $\lambda_6 = k \leq 310$.

Now we consider inequality (93). For $k \leq 310$ and $\ell \leq \min\{155, k - 1\}$, we compute the smallest value of $|\alpha^x 5^{k-\ell} (\alpha^k - 1)/(\alpha^\ell - 1) - 1|$, for an integer x. We get that this value is always $> 80/\alpha^{30}$. From inequality (93), we obtain that $(n-3)/10 \leq 30$, i.e. $n \leq 303$, a contradiction.

Hence, Theorem 1 is proved. Then we had a beer.

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