$D(n)$-QUINTUPLES WITH SQUARE ELEMENTS

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Abstract. For an integer n, a set of m distinct nonzero integers $\{a_1, a_2, \dots, a_m\}$ such that $a_i a_j + n$ is a perfect square for all $1 \le i < j \le m$, is called a $D(n)$-m-tuple. In this paper, we show that there are infinitely many essentially different $D(n)$-quintuples with square elements. We obtained this result by constructing genus one curves on a certain double cover of $\mathbb{A}^2$ branched along four curves.

1. Introduction

For an integer n, a set of m distinct nonzero integers with the property that the product of any two of its distinct elements plus n is a square is called a Diophantine m-tuple with the property $D(n)$ or $D(n)$-m-tuple. The $D(1)$-m-tuples (with rational elements) are called simply (rational) Diophantine m-tuples and have been studied since the ancient time.

The first example of a rational Diophantine quadruple was the set

$$\left\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\right\}$$

found by Diophantus. Fermat found the first Diophantine quadruple in integers $\{1, 3, 8, 120\}$. Euler proved that there exist infinitely many rational Diophantine quintuples (see [19]), in particular, he was able to extend the integer Diophantine quadruple found by Fermat to the rational quintuple

$$\left\{1, 3, 8, 120, \frac{777480}{8288641}\right\}.$$

Stoll [21] recently showed that this extension is unique.

In 1969, using linear forms in logarithms of algebraic numbers and a reduction method based on continued fractions, Baker and Davenport [3] proved that if d is a positive integer such that $\{1, 3, 8, d\}$ forms a Diophantine quadruple, then d has to be 120. This result motivated the conjecture that there does not exist a Diophantine quintuple in integers. The conjecture has been proved recently by He, Togbé and Ziegler [18] (see also [4, 8]).

On the other hand, it is not known how large can a rational Diophantine tuple be. In 1999, Gibbs found the first example of rational Diophantine sextuple [17]

$$\left\{\frac{11}{192}, \frac{35}{192}, \frac{155}{27}, \frac{512}{27}, \frac{1235}{48}, \frac{180873}{16}\right\}.$$

In 2017, Dujella, Kazalicki, Mikić and Szikszai [12] proved that there are infinitely many rational Diophantine sextuples, while Dujella and Kazalicki [10] (inspired by the work of Piezas [20]) described another construction of parametric families of rational Diophantine sextuples. Recently, Dujella, Kazalicki and Petričević in [14] proved that there are infinitely many rational Diophantine sextuples such that denominators of all the elements (in the lowest terms) in the sextuples are perfect

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squares, and in [13] they proved that there are infinitely many Diophantine sextuples containing two regular quadruples and one regular quintuple. No example of a rational Diophantine septuple is known. The Lang conjecture on varieties of general type implies that the number of elements of a rational Diophantine tuple is bounded by an absolute constant (see the introduction of [12]). Diophantine m-tuples have been studied over the rings other than $\mathbb{Z}$ and $\mathbb{Q}$, for example, Dujella and Kazalicki [11] computed the number of Diophantine quadruples over finite fields. For more information on Diophantine m-tuples see the survey article [9].

Sets with $D(n)$ properties have also been extensively studied. It is easy to show that there are no integer $D(n)$-quadruples if $n \equiv 2 \pmod 4$, and it is know that if $n \not\equiv 2 \pmod 4$ and $n \notin \{-4, -3, -1, 3, 5, 8, 12, 20\}$, then there is at least one $D(n)$-quadruple [6]. Recently, Bonciocat, Cipu and Mignotte [2] proved that there are no $D(-1)$-quadruples (as well as $D(-4)$-quadruples) thus leaving the existence of $D(n)$-quadruples in the remaining six sporadic cases open.

Dražić and Kazalicki [5] described rational $D(n)$-quadruples with a fixed product of elements in terms of points on certain elliptic curves. It is not known if there is a rational Diophantine $D(n)$-quintuple for every n, and no example of rational $D(n)$-sextuple is known if n is not a perfect square.

One can also study m-tuples that have $D(n)$-property for more than one n. Adžaga, Dujella, Kreso and Tadić [1] presented several families of Diophantine triples which have $D(n)$-property for two distinct n's with $n \neq 1$ as well as some Diophantine triples which are $D(n)$-sets for three distinct n's with $n \neq 1$. Dujella and Petričević in [15] proved that there are infinitely many (essentially different) integer quadruples which are simultaneously $D(n_1)$-quadruples and $D(n_2)$-quadruples with $n_1 \neq n_2$, and in [16] showed that the same thing is true for three distinct n's (since the elements of their quadruples are squares one of n's is equal to zero). Our main result extends the previous results to quintuples.

Note that if $\{a, b, c, d, e\}$ is a $D(n_1)$-quintuple, and u a nonzero rational, then $\{ua, ub, uc, ud, ue\}$ is a $D(n_1 u^2)$-quintuple and we say that these two quintuples are equivalent.

Theorem 1. *There are infinitely many nonequivalent quintuples that have $D(n_1)$ property for some $n_1 \in \mathbb{N}$ such that all the elements in the quintuple are perfect squares. In particular, there are infinitely many nonequivalent integer quintuples that are simultaneously $D(n_1)$-quintuples and $D(n_2)$-quintuples with $n_1 \neq n_2$ since then we can take $n_2 = 0$.*

Since every rational Diophantine quintuple is equivalent to some $D(u^2)$-quintuple whenever u is an integer divisible by the common denominator of the elements in the quintuple, Theorem 1 will follow if we prove that there are infinitely many rational Diophantine quintuples with the property that the product of any two of its elements is a perfect square.

A Diophantine quadruple $\{a, b, c, d\}$ is called regular if

$$(a + b - c - d)^2 = 4(ab + 1)(cd + 1).$$

Definition 1. We say that rational Diophantine quintuple $\{a, b, c, d, e\}$ is *exotic* if $abcd = 1$, quadruples $\{a, b, d, e\}$ and $\{a, c, d, e\}$ are regular, and if the product of any two of its elements is a perfect square.

Denote by S an affine surface defined over $\mathbb{Q}$ by

$$(1+r-2r^2t-t^2+rt^2)(-1+r+2r^2t+t^2+rt^2)(-r-r^2-2t-rt^2+r^2t^2)(r-r^2-2t+rt^2+r^2t^2) = y^2.$$

Define a rational map $p : S \to \mathbb{A}^5$ given by $p(r,t,y) = (a,b,c,d,e)$ where

$$a = \frac{(r^2-1)(t^2-1)(s^2-1)}{8rst},$$
$$b = \frac{2(t^2-1)rs}{(r^2-1)(s^2-1)t},$$
$$c = \frac{2(s^2-1)rt}{(r^2-1)(t^2-1)s},$$
$$d = \frac{2(r^2-1)st}{(s^2-1)(t^2-1)r},$$

where $s = \frac{-1+r^2+t+r^2t}{-1-r^2-t+r^2t}$ and e is defined by formula (1) from Section 2.

Proposition 2. *For every exotic quintuple $\{a,b,c,d,e\}$ there is a rational point $(r,t,y) \in S(\mathbb{Q})$ on the surface S such that $(a,b,c,d,e) = p(r,t,y)$. Conversely, if $(r,t,y) \in S(\mathbb{Q})$ is a rational point on S in the domain of p, then $p(r,t,y)$ is exotic quintuple provided that all elements are distinct and nonzero.*

Note that one can explicitly determine the degeneracy locus of map p – a finite set of curves on S such that for every $(r,t,y) \in S(\mathbb{Q})$ which is not on any of those curves we have that $p(r,t,y)$ is exotic quintuple. Thus, any curve on S with an infinite number of rational points will give rise to infinitely many exotic quintuples.

Denote by $\pi : S \to \mathbb{A}^2$ the projection $\pi(r,t,y) = (r,t)$. Let D_1, D_2 and D_3 be plane genus zero curves in $\mathbb{A}^2$ defined by

$$D_1 : r^2t^2 - 4r^2t - 3r^2 - 2rt^2 - 2r - 3t^2 - 4t + 1 = 0,$$
$$D_2 : r^2t - r^2 + 2rt^2 + 2r - t - 1 = 0,$$
$$D_3 : r^2t^2 + 3r^2 - t^2 + 2t - 1 = 0,$$

and $\widetilde{D_i} = \pi^{-1}(D_i)$ pullbacks of these curves to S via π.

Proposition 3. *Curves $\widetilde{D_1}, \widetilde{D_2}$ and $\widetilde{D_3}$ are genus one curves defined over $\mathbb{Q}$ birationally equivalent to elliptic curves with positive Mordell-Weil rank. In particular, there are infinitely many exotic quintuples.*

Remark 1. The surface S, as an affine subvariety of a double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ ramified in four curves of type $(2,2)$, is of general type. According to the Bombieri-Lang Conjecture, there should be only finitely many curves of geometric rank 0 or 1 on S (and they should account for all but finitely many rational points of S). It is an interesting question if there is any other genus 0 or 1 curve on S besides the 20 genus 0 curves in degeneracy locus, four ramification curves (which have genus 1) and three curves $\widetilde{D_i}$.

For an example, consider the following parametrization of D_3

$$(r,t) = \left(-\frac{2u+1}{u^2+u+1}, \frac{u^2+4u+1}{(u-1)(u+1)}\right).$$

It defines a curve birational to $\widetilde{D_3}$ given by quartic

$$-48\left(u^2-3u-1\right)\left(u^2+5u+3\right) = v^2.$$

The point $(u,v) = (3,36)$ of this quartic corresponds to $(r,t) = (-\frac{7}{13}, \frac{11}{4})$ which in turn is mapped by p to the Diophantine quintuple

$$M = \left\{\frac{225^2}{480480}, \frac{2548^2}{480480}, \frac{286^2}{480480}, \frac{1408^2}{480480}, \frac{819^2}{480480}\right\}.$$

Our investigation began with this quintuple.

2. Parametrizing exotic quintuples

Our starting point was the experimental discovery of an exotic rational Diophantine quintuple M (defined in the introduction) which by clearing denominators gives Diophantine $D(480480^2)$-quintuple with square elements. This quintuple $\{a, b, c, d, e\}$ has the following structure which motivated our construction of infinite families

i) $abcd = 1$,
ii) quadruples $\{a, b, d, e\}$ and $\{a, c, d, e\}$ are regular.

Proposition 4. *Let $\{a, b, c, d\}$ be a rational Diophantine quadruple with $abcd = 1$. Then there exist $r, s, t \in \mathbb{Q}\backslash\{-1, 0, 1\}$ such that*

$$a = xyz, \quad b = \frac{x}{yz}, \quad c = \frac{y}{xz}, \quad d = \frac{z}{xy},$$

where $x = \frac{t^2-1}{2t}, y = \frac{s^2-1}{2s}$ and $z = \frac{r^2-1}{2r}$. In particular, the product of any two elements of the quadruple is a perfect square.

Proof. From $ab + 1 = ab + abcd = ab(1 + cd)$ it follows that ab is a perfect square, and similarly for other pairwise products. Set $ab = x^2, ac = y^2$ and $ad = z^2$, with $x, y, z \in \mathbb{Q}$. It follows $a^2 = \frac{ab\cdot ac}{bc} = \frac{x^2y^2}{1/z^2}$, hence $a = xyz$ and $b = \frac{x}{yz}$, $c = \frac{y}{xz}$ and $d = \frac{z}{xy}$ (with the appropriate choice of signs). Since x^2+1 is a perfect square, there is $t \in \mathbb{Q}$ such that $x = \frac{t^2-1}{2t}$, and similarly for y and z. The claim follows. $\square$

To extend quadruple $\{a, b, c, d\}$ defined by $r, s, t \in \mathbb{Q}$ (as in Proposition 4) to an exotic quintuple it is enough that triples $\{a, b, d\}$ and $\{a, c, d\}$ have a common regular extension e such that ae is a perfect square.

Since both $\{a, b, d\}$ and $\{a, c, d\}$ extend to regular quadruples in two different ways, $\{e_1, e_1'\}$ and $\{e_2, e_2'\}$ respectively, to check if there is a common regular extension a priori we have four conditions to inspect. It is easy to see that the maps $\sigma_1(r, s, t) = (1/r, s, 1/t)$ and $\sigma_2(r, s, t) = (\frac{1}{r}, s, -t)$ are symmetries of the equations from Proposition 4 defining a, b, c and d, hence both (r, s, t) and $\sigma_i(r, s, t)$ give rise to the same quadruple (a, b, c, d). In general the map $(r, s, t) \mapsto (a, b, c, d)$ is $32 : 1$, but we will not need the whole group of symmetries. Moreover, σ_1 "maps" e_2 to e_2' and fixes e_1, while σ_2 maps e_1 to e_1' and fixes e_2. Therefore, to parametrize quadruples with common regular extension as above it is enough to solve $e_1 = e_2$ for any choice of e_1 and e_2. Thus for the choice of e_1 and e_2

$$e_1 = \frac{u_1(r, s, t)u_2(r, s, t)u_3(r, s, t)u_4(r, s, t)}{8(-1 + r)r(1 + r)(-1 + s)s(1 + s)(-1 + t)t(1 + t)}, \tag{1}$$

where

$$u_1(r, s, t) = -1 - r + s - rs - t - rt - st + rst,$$
$$u_2(r, s, t) = 1 + r - s + rs - t - rt - st + rst,$$
$$u_3(r, s, t) = 1 - r - s - rs + t - rt + st + rst,$$
$$u_4(r, s, t) = -1 + r + s + rs + t - rt + st + rst,$$

and

$$e_2 = \frac{v_1(r, s, t)v_2(r, s, t)v_3(r, s, t)v_4(r, s, t)}{8(-1 + r)r(1 + r)(-1 + s)s(1 + s)(-1 + t)t(1 + t)},$$

where

$$
\begin{aligned}
v_1(r,s,t) &= -1-r-s-rs+t-rt-st+rst,\\
v_2(r,s,t) &= 1+r-s-rs-t+rt-st+rst,\\
v_3(r,s,t) &= 1-r+s-rs-t-rt+st+rst,\\
v_4(r,s,t) &= -1+r+s-rs+t+rt+st+rst,
\end{aligned}
$$

we obtain the following condition

$$(s-t)(1+st)(1-r^2-s-r^2s-t-r^2t-st+r^2st)(-1-r^2+s-r^2s+t-r^2t+st+r^2st) = 0.$$

Solutions to $(s-t)(1+st) = 0$ induce degenerate quintuples (with zero element or with two identical elements) so we can ignore them. To reduce the argument further, note that a, b, c, d, e_1 and e_2 are fixed by the map $\sigma_3(r,s,t) = (-1/r, s, -1/t)$. Moreover, σ_3 defines a birational map between affine plane surfaces defined by $1-r^2-s-r^2s-t-r^2t-st+r^2st = 0$ and $-1-r^2+s-r^2s+t-r^2t+st+r^2st = 0$ which is an isomorphism outside the vanishing set of $rst(r^2-1)(s^2-1)(t^2-1) = 0$. Since the triples (r,s,t) from this vanishing set do not correspond to Diophantine quintuples, without loss of generality we can assume that the triples (r,s,t) describing rational Diophantine quintuples $\{a,b,c,d,e\}$, such that $abcd = 1$ and that both $\{a,b,d,e\}$ and $\{a,c,d,e\}$ are regular, satisfy

$$1-r^2-s-r^2s-t-r^2t-st+r^2st = 0. \tag{2}$$

On the other hand, the condition that ae_1 is a perfect square is equivalent to

$$
\begin{aligned}
&(-1-r+s-rs-t-rt-st+rst)(1+r-s+rs-t-rt-st+rst)\\
&(1-r-s-rs+t-rt+st+rst)(-1+r+s+rs+t-rt+st+rst) = y^2.
\end{aligned}
$$

Substituting s from (2) we obtain a defining equation for the affine surface S defined in the introduction. Thus, we have constructed the rational map from the introduction

$$p : S \to \mathbb{A}^5, \quad p(r,t,y) = (a,b,c,d,e),$$

(defined by (1), (2) and Proposition 4) and proved that for every exotic quintuple (a,b,c,d,e) there is a rational point (r,t,y) on the surface S such that $p(r,t,y) = (a,b,c,d,e)$. Note that the pair (r,t) defining the point is not necessarily unique.

Conversely, given $(r,t,y) \in S(\mathbb{Q})$ such that $p(r,t,y)$ is defined, the quintuple $p(r,t,y)$ will be exotic if it is non-degenerate (all elements must be distinct and nonzero). This finishes the proof of Proposition 2.

3. Construction of curves on S

If we show that the surface S has infinitely many rational points outside the degeneracy locus of p (a finite set of curves on S whose rational points either map under p to degenerate quintuples or p is not defined for them), then Proposition 2 will imply Theorem 1. For that, we will construct genus one curves on S which are defined over $\mathbb{Q}$ and birationaly equivalent to elliptic curves (over $\mathbb{Q}$) of positive Mordell-Weil rank.

Denote by $\pi : S \to \mathbb{A}^2$ the projection $\pi(r,t,y) = (r,t)$, and denote by

$$
\begin{aligned}
C_1 &: 1+r-2r^2t-t^2+rt^2 = 0,\\
C_2 &: -1+r+2r^2t+t^2+rt^2 = 0,\\
C_3 &: -r-r^2-2t-rt^2+r^2t^2 = 0,\\
C_4 &: r-r^2-2t+rt^2+r^2t^2 = 0,
\end{aligned}
$$

curves over which the map π is ramified.

The configuration of these curves has a large symmetry group. One can readily check that the maps

$$\tau_1(r,t) = (-r,t), \quad \tau_2(r,t) = \left(\frac{1}{r}, \frac{1}{t}\right),$$
$$\tau_3(r,t) = \left(-r, \frac{t-1}{t+1}\right),$$

extend to birational automorphisms of $C_1 \cup C_2 \cup C_3 \cup C_4$, and also to birational automorphisms of S. While we have already encountered maps τ_1 and τ_2, note that if $p(r,t) = (a,b,c,d,e)$, then $p(\tau_3(r,t)) = (-d,-c,-b,-a,-e)$. They generate a group G of order 16 which acts on the set of plane curves $D \subset \mathbb{A}^2$. We say that two plane curves D_1 and D_2 are equivalent if there is $\tau \in G$ such that it generates birational map from D_1 to D_2.

Remark 2. We remark one curiosity related to τ_3. Note that if (a,b,c,d) is rational Diophantine quadruple with $abcd = 1$ which corresponds to the triple (r,s,t), then rational Diophantine quadruple $(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d})$ corresponds to the triple $(\frac{r-1}{r+1}, \frac{s-1}{s+1}, \frac{t+1}{t-1})$.

Let $D \subset \mathbb{A}^2$ be a plane curve of genus zero, and denote by $\widetilde{D} = \pi^{-1}(D)$ a pullback of D under π. Assume that $\widetilde{D}$ is absolutely irreducible of genus g. Genus g is controlled by the ramification of $\pi|\widetilde{D}$. More precisely, if we resolve singularities of the projective closures of $\widetilde{D}$ and D, and apply Riemman-Hurwitz formula to the corresponding extension $\widetilde{\pi}$ of $\pi|\widetilde{D}$ we will get

$$2g - 2 = -4 + N,$$

where N is the number of ramification points of $\widetilde{\pi}$. In particular, if we want g to be one, then N must be equal to four.

Denote $L = \bigcup_{i \neq j} C_i \cap C_j$. We have the following criterion for ramification points of $\widetilde{\pi}$.

Lemma 5. *Assume that, for some i, C_i and D intersect transversally at P. If $P \notin L$ and if P is nonsingular on D, then $\widetilde{\pi}$ is ramified at P.*

The previous lemma suggests that if we want to search for a genus zero plane curve D for which $\widetilde{D} = \pi^{-1}(D)$ is genus one curve, our best candidates would be curves that intersect $\cup_i C_i$ outside L in as few points as possible. This task gets harder as the degree of D gets bigger — by Bézout's theorem D and C_i intersect at $3 \deg D$ or $4 \deg D$ points (counting multiplicities and points at infinity). Also, to control the genus of D one needs to specify singularities (whose number is described by Plücker's formula) which a priori can be anywhere (but it works best for us if they are on $\cup C_i$ since then this intersection will probably not count for ramification) so this made systematic computer search impossible for us to implement.

In addition to this approach, in order to employ the symmetry group G, we also searched for curves D on which some $\tau \in G$ induces birational automorphism. The logic behind this is that if, for example, such D intersects C_i (ideally) in L, and if τ induced birational map between C_i and C_j, then D also intersects C_j in L (since τ maps L into itself).

4. Results

In our computer search we found three inequivalent genus zero curves $D_1, D_2, D_3 \subset \mathbb{A}^2$ defined in the introduction such that the curves $\widetilde{D_i} = \pi^{-1}(D_i)$ are genus one curves birational to elliptic curve with positive Mordell-Weil rank. Interestingly, the orbit of each of these curves under the action of G is of size 8 — the curves are fixed by elements $\tau_2, \tau_2 \circ \tau_3^2$ and τ_1, respectively. The following analysis of curves $\widetilde{D_i}$ finishes the proof of Proposition 3.

4.1. **Curve $\widetilde{D_1}$.** We have the following parametrization of the curve D_1

$$\psi_1(u) = \left(-\frac{3u^2+4u-1}{3u^2+8u+3}, \frac{2u}{(u+1)(3u+1)}\right),$$

which gives the following model for $\widetilde{D_1}$

$$-192\left(5u^2+10u+3\right)\left(9u^2+18u-1\right) = v^2,$$

which is birational to the elliptic curve

$$E_1 : y^2 = x^3 - 2892x - 59024,$$

of rank 1 and torsion subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. A generator of infinite order in $E_1(\mathbb{Q})$ corresponds to $u = -60/233$ and parameterizes the following exotic quintuple

$$a = -\frac{2952994011087867871 7653}{420081952495961042800800}, \quad b = -\frac{3041992513146972959}{115488479640779256},$$
$$c = -\frac{351416293757343837}{2249352029178441082}, \quad d = -\frac{1776863948138083954777600}{5140041910127682086 30559},$$
$$e = -\frac{9276432833615399134 82847}{141804790226710724159200}.$$

4.2. **Curve $\widetilde{D_2}$.** We have the following parametrization of the curve D_2

$$\psi_2(u) = \left(\frac{u(2u+1)}{(u+1)(u+2)}, -\frac{u^2-2u-2}{u(u+2)}\right),$$

which gives the following model for $\widetilde{D_2}$

$$48\left(u^2+16u+10\right)\left(3u^2-2\right) = v^2,$$

which is birational to the elliptic curve

$$E_2 : y^2 = x^3 - 876x - 9520,$$

of rank 1 and torsion subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. A generator of infinite order in $E_2(\mathbb{Q})$ corresponds to $u = \frac{113}{23}$ and parameterizes the following exotic quintuple

$$\left\{-\frac{482493852225}{293535838544}, -\frac{1058592509345792}{1212259417081713}, -\frac{1207470487056449}{74793264945984}, -\frac{18858398366873}{437001310622800}, -\frac{695331110026639}{116388239242275}\right\}.$$

4.3. **Curve $\widetilde{D_3}$.** We have the following parametrization of the curve D_3

$$\psi_3(u) = \left(-\frac{2u+1}{u^2+u+1}, \frac{u^2+4u+1}{(u-1)(u+1)}\right),$$

which gives the following model for $\widetilde{D_3}$

$$-48\left(u^2-3u-1\right)\left(u^2+5u+3\right) = v^2,$$

which is birational to the elliptic curve

$$E_3 : y^2 + xy + y = x^3 - x^2 - 41x + 96,$$

of rank 1 and torsion subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. A generator of infinite order in $E_3(\mathbb{Q})$ corresponds to $u = -4$ and parameterizes the following exotic quintuple

$$\left\{-\frac{5632}{1365}, -\frac{4459}{330}, -\frac{143}{840}, -\frac{3375}{32032}, -\frac{2457}{1760}\right\}.$$

5. Concluding remarks

While we have found infinitely many rational Diophantine quintuples with $D(0)$ property, it remains open if there is a rational Diophantine quintuple with square elements. On the other hand, there are infinitely many rational Diophantine quadruples with square elements, for example the following two parameter family has this property

$$\begin{aligned}
a &= \frac{3^2(s-1)^2(s+1)^2v^2}{2^2(2s^3-2s+v^2)^2},\\
b &= \frac{v^2(-4s^3+4s+v^2)^2}{2^2(s+1)^2(s-1)^2(-s^3+s+v^2)^2},\\
c &= \frac{(2s^3-2s+v^2)^2}{3^2v^2s^2},\\
d &= \frac{4^2(-s^3+s+v^2)^2s^2}{v^2(-4s^3+4s+v^2)^2}.
\end{aligned}$$

This family is obtained by taking $t = 1/(r-1)$ in the notation of Proposition 4. We have also found an example of a rational Diophantine quadruple with square elements for which the product $abcd \neq 1$

$$\left\{\left(\frac{18}{77}\right)^2, \left(\frac{55}{96}\right)^2, \left(\frac{56}{15}\right)^2, \left(\frac{340}{77}\right)^2\right\}.$$

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