EXTENSIONS OF A DIOPHANTINE TRIPLE BY ADJOINING SMALLER ELEMENTS II

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Abstract. Let $\{a_1, b, c\}$ and $\{a_2, b, c\}$ be Diophantine triples with $a_1 < b < a_2 < c$ and $a_2 \neq b + c - 2\sqrt{bc+1}$. Put $d_2 = a_2 + b + c + 2a_2bc - 2r_2st$, where $r_2 = \sqrt{a_2b+1}$, $s = \sqrt{ac+1}$ and $t = \sqrt{bc+1}$. In this paper, we prove that if $c \leq 16\mu^2b^3$, where $\mu = \min\{a_1, d_2\}$, then $\{a_1, a_2, b, c\}$ is a Diophantine quadruple. Combining this result with one of our previous results implies that if $\{a_i, b, c, d\}$ $(i \in \{1, 2, 3\})$ are Diophantine quadruples with $a_1 < a_2 < b < a_3 < c < d$, then $a_3 = b + c - 2\sqrt{bc+1}$. It immediately follows that there does not exist a septuple $\{a_1, a_2, a_3, a_4, b, c, d\}$ with $a_1 < a_2 < b < a_3 < a_4 < c < d$ such that $\{a_i, b, c, d\}$ $(i \in \{1, 2, 3, 4\})$ are Diophantine quadruples. Moreover, it is shown that there are only finitely many sextuples $\{a_1, a_2, a_3, b, c, d\}$ with $a_1 < b < a_2 < a_3 < c < d$ such that $\{a_i, b, c, d\}$ $(i \in \{1, 2, 3\})$ are Diophantine quadruples.

1. Introduction

A Diophantine m-tuple is defined as a set $\{a_1, \dots, a_m\}$ of m distinct positive integers satisfying the property that $a_ia_j + 1$ is a perfect square for all i and j with $1 \leq i < j \leq m$. It is easy to find examples of Diophantine pairs, such as $\{1, 3\}$, $\{2, 12\}$, $\{K, K + 2\}$ for a positive integer K and $\{F_{2n}, F_{2n+2}\}$ with F_k being the kth Fibonacci number. For a fixed Diophantine pair $\{a, b\}$, Euler found that $\{a, b, a + b + 2r\}$ is a Diophantine triple, where $r = \sqrt{ab+1}$. Such a Diophantine triple is called regular, and it is known that the largest element $c = a + b + 2r$ is minimal among all the possible c's such that $\{a, b, c\}$ is a Diophantine triple with $c > \max\{a, b\}$. Note that $c = a + b + 2r$ is equivalent to $b = a + c - 2\sqrt{ac+1}$ and to $a = b + c - 2\sqrt{bc+1}$.

Let $\{a, b, c\}$ be a Diophantine triple and r, s, t the positive integers satisfying $ab + 1 = r^2$, $ac + 1 = s^2$, $bc + 1 = t^2$. Define

$$d_+ := d_+(a, b, c) = a + b + c + 2abc + 2rst,$$
$$d_- := d_-(a, b, c) = a + b + c + 2abc - 2rst.$$

It was found in [1] and [9], independently, that $\{a, b, c, d_+\}$ always forms a Diophantine quadruple. Such a quadruple is called regular. Note that $0 \leq d_- < c$ and that $d_- \geq 1$ if and only if $c \neq a + b + 2r$. Thus, if $c \neq a + b + 2r$, then $\{a, b, c, d_-\}$ is also regular with $c = d_+(a, b, d_-)$. The following conjecture posed in [1] and [9] is still open.

Conjecture 1.1. *Any Diophantine quadruple is regular.*

In the previous work [3], the authors closely examined when $\{a_1, b, c, d\}$ and $\{a_2, b, c, d\}$ with $a_1 < a_2 < \min\{b, c, d\}$ can be Diophantine quadruples, and showed

2020 *Mathematics Subject Classification.* 11D09, 11D45, 11B37, 11J68.

Key words and phrases. Diophantine m-tuples and Pellian equations.

A. D. acknowledges support from the QuantiXLie Center of Excellence, a project cofinanced by the Croatian Government and European Union through the European Regional Development Fund — the Competitiveness and Cohesion Operational Programme (Grant KK.01.1.1.01.0004) and the Croatian Science Foundation under the project no. IP-2018-01-1313.

that for a fixed Diophantine triple $\{b, c, d\}$ there exist at most two positive integers a with $a < \min\{b, c, d\}$ such that $\{a, b, c, d\}$ is a Diophantine quadruple. The main purpose of this paper is to show a result analogous to the above in the case where not all a's are smaller than $\min\{b, c, d\}$. To do this, we prove the following.

Theorem 1.2. *Let $\{a_1, b, c\}$ and $\{a_2, b, c\}$ be Diophantine triples with $a_1 < b < a_2 < c$ and $a_2 \neq b + c - 2\sqrt{bc+1}$. If $c \leq 16\mu^2 b^3$, where $\mu = \min\{a_1, d_2\}$ and $d_2 = d_-(a_2, b, c)$, then $\{a_1, a_2, b, c\}$ is a Diophantine quadruple.*

Note that Theorem 1.2 rectifies [8, Theorem 1.6], which asserts the following:

Let $\{a_1, b, c\}$ and $\{a_2, b, c\}$ be Diophantine triples with

$$a_1 < \min\{a_2, b\}, \quad \max\{a_2, b\} < c < 16b^3.$$

Then, $\{a_1, a_2, b, c\}$ is a Diophantine quadruple.

In the case where $a_2 < b$, this is nothing but [2, Theorem 1.4] and certainly true. However, in the case where $b < a_2$, this is incorrect, as easily seen, e.g., from the counterexample with $a_1 = 1$, $b = 3$, $a_2 = 85$, $c = 120$. The incorrectness of [8, Theorem 1.6] has caused that of [8, Theorem 1.7]. On this occasion, we replace it with an assertion shown by a correct argument. Denote by $Q(a, b)$ the number of irregular Diophantine quadruples containing a fixed pair $\{a, b\}$.

Theorem 1.3. *Let $\{a, b\}$ be a Diophantine pair with*

$$a < b \leq 4a^3 + 16a^2 + 24a + 16. \tag{1.1}$$

Then, $Q(a, b) \leq 122$.

Theorem 1.2 together with [3, Main Theorem] implies the following.

Corollary 1.4. *Let $\{a_i, b, c, d\}$ $(i \in \{1, 2, 3\})$ be Diophantine quadruples with $a_1 < a_2 < b < a_3 < c < d$. Then, $a_3 = b + c - 2\sqrt{bc+1}$.*

The following corollary is a direct consequence of Corollary 1.4.

Corollary 1.5. *There does not exist a septuple $\{a_1, a_2, a_3, a_4, b, c, d\}$ with $a_1 < a_2 < b < a_3 < a_4 < c < d$ such that $\{a_i, b, c, d\}$ $(i \in \{1, 2, 3, 4\})$ are Diophantine quadruples.*

Analogously to [3, Corollary 1.5], Theorem 1.2 also implies the following finiteness results.

Corollary 1.6. *There are only finitely many sextuples $\{a_1, a_2, a_3, b, c, d\}$ with $a_1 < b < a_2 < a_3 < c < d$ such that $\{a_i, b, c, d\}$ $(i \in \{1, 2, 3\})$ are Diophantine quadruples.*

Corollary 1.7. *There are only finitely many quintuples $\{a_1, a_2, b, c, d\}$ with $a_1 < b < a_2 < c < d$ such that $\{a_i, b, c, d\}$ $(i \in \{1, 2\})$ are Diophantine quadruples and $a_2 \neq b + c - 2\sqrt{bc+1}$.*

The forthcoming section is devoted to proving Theorems 1.2 and 1.3. In Section 3, the proofs of Corollaries 1.4, 1.5, and 1.7 are given in turn.

2. Proofs of theorems

We start by pointing out a result that should be better known.

Lemma 2.1. *Let $\{A, B, C\}$ be a Diophantine triple with $A < B < C$ and R, S, T the positive integers defined by $AB + 1 = R^2$, $AC + 1 = S^2$, $BC + 1 = T^2$. If $d_-(A, B, C) > B$ then $C > 4AB^2 + 4B + 2A$.*

Proof. The assumption $d_-(A,B,C) > B$ is equivalent to $A+C+2ABC > 2RST$. Squaring this inequality, one finds $C^2 - 2(2AB^2+2B+A)C + A^2 - 4AB - 4 > 0$, whence the desired conclusion. $\square$

Proof of Theorem 1.2. Put $d_1 = d_-(a_1,b,c)$ and $d_2 = d_-(a_2,b,c)$. Suppose that $c \le 16\mu^2 b^3$ with $\mu = \min\{a_1,d_2\}$. As is well-known (see, e.g., [7, Introduction]), it holds that

$$(b+c-a_i-d_i)^2 = 4(a_id_i+1)(bc+1) \tag{2.1}$$

and

$$c = 4a_id_ib + \lambda_i \max\{d_i,a_i,b\} \tag{2.2}$$

for $i \in \{1,2\}$ with λ_i a rational number satisfying $1 < \lambda_i < 4$. Equality (2.2) implies

$$|a_1d_1 - a_2d_2| < \frac{\max\{d_1,d_2,a_2\}}{b}. \tag{2.3}$$

If $a_1d_1 = a_2d_2$, then from (2.1) one has

$$(b+c-a_1-d_1)^2 = (b+c-a_2-d_2)^2.$$

It follows from the proof of [8, Theorem 1.6] that $d_1 = a_2$ and $d_2 = a_1$, which yield that $\{a_1,a_2,b,c\}$ is a Diophantine quadruple.

Now, assume that $a_1d_1 \ne a_2d_2$. From this, we will derive a contradiction. If $\max\{d_1,d_2,a_2\} = d_1$, then by (2.2) and Lemma 2.1,

$$c > 4a_1d_1b > 16a_1b^2(a_1b+1) > 16a_1^2b^3,$$

which contradicts the assumption; if $\max\{d_1,d_2,a_2\} = d_2$, then by (2.2) and Lemma 2.1,

$$c > 4a_2d_2b > 16a_2b^2(a_2b+1) > 16a_2^2b^3,$$

which is a contradiction, too. It therefore remains to consider the case where $\max\{d_1,d_2,a_2\} = a_2$. Note that the proof of [8, Theorem 1.6] neglected to take this possibility into consideration. In this case, since

$$|a_1d_1 - a_2d_2| = |x_1^2 - x_2^2| \ge |(x_2-1)^2 - x_2^2| = 2x_2 - 1,$$

where $x_i = \sqrt{a_id_i+1}$ for $i \in \{1,2\}$, one sees from (2.3) that

$$2x_2 < \frac{a_2}{b} + 1.$$

Squaring both sides of this inequality, one gets

$$a_2^2 - 2b(2d_2b-1)a_2 - 3b^2 > 0,$$

which yields $a_2 > 2b(2d_2b-1)$ and $x_2 > 2d_2b - 1$. If $d_2 \ge 2$, then $x_2 \ge 2d_2b+1$, since $\gcd(x_2,d_2) = 1$. If $d_2 = 1$ and $x_2 = 2b$, then $a_2 = x_2^2 - 1 = 4b^2 - 1$. It follows from (2.2) that

$$c > 4a_2d_2b + a_2 = 4b(4b^2-1) + 4b^2 - 1 > 16b^3,$$

which contradicts the assumption $c \le 16\mu^2b^3$. Noting that the hypothesis $a_2 \ne b + c - 2\sqrt{bc+1}$ implies $d_2 \ge 1$, we thus deduce that $x_2 \ge 2d_2b+1$ in any case. Therefore, from (2.2) we obtain

$$c > 4a_2d_2b \ge 16d_2b^2(d_2b+1) > 16d_2^2b^3,$$

which is a contradiction again. This completes the proof of Theorem 1.2. $\square$

In order to prove Theorem 1.3, we need a corrected version of [8, Proposition 4.2], which requires some notation.

Let $\{a,b,c\}$ be a Diophantine triple with $a<b$ and r, s, t the positive integers satisfying

$$ab+1=r^2, \quad ac+1=s^2, \quad bc+1=t^2.$$

Eliminating c from the last two equations above, we obtain the Pellian equation

$$at^2-bs^2=a-b. \tag{2.4}$$

By Nagell's argument (see [11, Theorem 108a] and [6, Lemma 1]), one sees that for any positive solution (t,s) to (2.4) there exists a solution (t_0,s_0) to (2.4) and a non-negative integer ν such that

$$t\sqrt{a}+s\sqrt{b}=(t_0\sqrt{a}+s_0\sqrt{b}\,)(r+\sqrt{ab}\,)^{\nu} \tag{2.5}$$

with

$$0<|t_0|\le\sqrt{\frac{(r-1)(b-a)}{2a}}, \quad 0<s_0\le\sqrt{\frac{a(b-a)}{2(r-1)}}. \tag{2.6}$$

Equation (2.5) enables us to write $s=\sigma_\nu^\lambda$ with $\lambda\in\{\pm\}$, where

$$\sigma_0:=\sigma_0^\lambda=s_0, \;\; \sigma_1^\lambda=rs_0+\lambda a|t_0|, \;\; \sigma_{\nu+2}^\lambda=2r\sigma_{\nu+1}^\lambda-\sigma_\nu^\lambda, \tag{2.7}$$

and $t_0=\lambda|t_0|$. In case $(t_0,s_0)=(\pm1,1)$, put

$$s_\nu^\lambda=\sigma_\nu^\lambda \;\text{ and }\; c_\nu^\lambda=\frac{(s_\nu^\lambda)^2-1}{a}.$$

Then, $\{a,b,c_\nu^\lambda\}$ is a Diophantine triple for any $\nu\ge1$ and $\lambda\in\{\pm\}$.

The following proposition is the corrected version of [8, Proposition 4.2].

Proposition 2.2. *Let $\{a,b\}$ be a Diophantine pair with* (1.1). *Then, there exist at most two distinct solutions (t_0,s_0) and (t_0',s_0') to* (2.4) *satisfying* (2.6) *and $(t_0,s_0)\ne(\pm1,1)\ne(t_0',s_0')$ such that for any Diophantine triple $\{a,b,c\}$ it holds $c\in\{c_\nu^\lambda,\alpha_\nu^\lambda,\beta_\nu^\lambda\}$ for some ν and λ, where $\{\alpha_\nu^\lambda\}$ and $\{\beta_\nu^\lambda\}$ are the sequences defined by*

$$\alpha_\nu^\lambda=\frac{(\sigma_\nu^\lambda)^2-1}{a} \;\;\textit{and}\;\; \beta_\nu^\lambda=\frac{((\sigma_\nu')^\lambda)^2-1}{a},$$

where $\{\sigma_\nu^\lambda\}$ and $\{(\sigma_\nu')^\lambda\}$ are the recurrent sequences given by (2.7) *and*

$$(\sigma_0')^\lambda=s_0', \;\; (\sigma_1')^\lambda=rs_0'+\lambda a|t_0'|, \;\; (\sigma_{\nu+2}')^\lambda=2r(\sigma_{\nu+1}')^\lambda-(\sigma_\nu')^\lambda$$

with $t_0'=\lambda|t_0'|$, respectively.

Proof. Note that the assumption (1.1) entails $r^2=ab+1<4(a+1)^4$. Hence, $r\le 2a^2+4a+1$, which implies

$$a<b\le 4a^3+16a^2+20a+8. \tag{2.8}$$

Thus, the hypothesis of Theorem 1.3 from [8] is satisfied.

Suppose that we need a solution $(t_0'',s_0'')\notin\{(\pm1,1),(\pm t_0,s_0),(\pm t_0',s_0')\}$ to (2.4) satisfying (2.6) to express the third element c in a Diophantine triple $\{a,b,c\}$. Denote by $\{\gamma_\nu^\lambda\}$ the sequence defined by $\gamma_\nu^\lambda=(((\sigma_\nu'')^\lambda)^2-1)/a$, where

$$(\sigma_0'')^\lambda=s_0'', \;\; (\sigma_1'')^\lambda=rs_0''+\lambda a|t_0''|, \;\; (\sigma_{\nu+2}'')^\lambda=2r(\sigma_{\nu+1}'')^\lambda-(\sigma_\nu'')^\lambda$$

with $t_0''=\lambda|t_0''|$. We know from [8, Lemma 4.1] that

$$1\le\alpha<a, \quad 1\le\beta<a, \quad 1\le\gamma<a$$

for some $\alpha\in\{\alpha_0^\lambda,\alpha_1^-\}$, $\beta\in\{\beta_0^\lambda,\beta_1^-\}$, $\gamma\in\{\gamma_0^\lambda,\gamma_1^-\}$. As noted in the proof of [8, Proposition 4.2], we may assume that $a\ge2$. When $a=2$, inequality (2.8) gives

$b \le 144$, so that there are eight values for b such that $\{2, b\}$ is a Diophantine pair. From inequality (2.6) one then sees that $0 < s_0 < \sqrt{(144-2)/(17-1)} < 3$. Hence, there are two possible values for s_0, whence the desired conclusion follows.

For $a \ge 3$ one has $b \le 4a^3+16a^2+20a+8 < 16a^3$. It follows from [2, Theorem 1.4] that if α, β, γ are different from one another, then $\{a, b, \alpha, \beta\}$, $\{a, b, \alpha, \gamma\}$, $\{a, b, \beta, \gamma\}$ are Diophantine quadruples, and thus $\{a, b, \alpha, \beta, \gamma\}$ is a Diophantine quintuple, which contradicts [10, Theorem 1]. Since it is easy to prove that $\alpha \ne \beta \ne \gamma \ne \alpha$ (see the proof of [8, Proposition 4.2]), this completes the proof of Proposition 2.2. □

It is also necessary to repair the assertion of [8, Proposition 4.3] as follows.

Proposition 2.3. *Suppose that $\{a, b, c, d\}$ is an irregular Diophantine quadruple with* (1.1) *and $c < d$. Let $\{\alpha_\nu^\lambda\}$ and $\{\beta_\nu^\lambda\}$ be the sequences appearing in Proposition* 2.2. *Then, $c \le \max\{c_3^+, \alpha_3^+, \beta_3^+\}$.*

Proof. This immediately follows as in the proof of [8, Proposition 4.3]. □

Now, we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. We may assume that $c < d$. Proposition 2.3 implies that

$$c \in \{c_\nu^\lambda, \alpha_0, \alpha_\nu^\lambda, \beta_0, \beta_\nu^\lambda\}$$

for some $\nu \in \{1, 2, 3\}$ and $\lambda \in \{\pm\}$. We only have to consider the cases where

$$c \in \{\beta_0, \beta_1^-, \beta_1^+, \beta_2^-, \beta_2^+, \beta_3^-, \beta_3^+\},$$

besides the remaining 13 cases taken into account in the proof of [8, Theorem 1.7]. It follows from that proof that

$$Q(a, b) \le 3 \times 2 + 6 \times 4 + (6 \times 3 + 7 \times 4) \times 2 = 122.$$

□

3. Proofs of the corollaries

Proof of Corollary 1.4. Assume that $a_3 \ne b+c-2\sqrt{bc+1}$. Then, Theorem 1.2 applied for $\{a_2, b, c\}$ and $\{a_3, b, c\}$ implies that if $c \le 16(\mu')^2b^3$, where $\mu' = \min\{a_2, d_3\}$ and $d_3 = d_-(a_3, b, c)$, then $\{a_2, a_3, b, c, d\}$ would be a Diophantine quintuple, which contradicts [10, Theorem 1]. Thus, $c > 16(\mu')^2b^3$. Since $a_3 \ne b + c - 2\sqrt{bc+1}$, it holds that $d_3 \ge 1$. Moreover, if $d_3 \in \{a_1, a_2\}$, then $\{d_3, a_3, b, c, d\}$ would be a Diophantine quintuple, which again contradicts [10, Theorem 1]. It follows from [3, Corollary 1.4] that $d_3 > b$. However, we then have $c > 16(\mu')^2b^3 = 16a_2^2b^3$, which contradicts [3, Main Theorem]. □

Proof of Corollary 1.6. Assume that $\{a_i, b, c, d\}$ $(i \in \{1, 2, 3\})$ are Diophantine quadruples with $a_1 < b < a_2 < a_3 < c < d$. Since both a_2 and a_3 cannot simultaneously equal $c+d-2\sqrt{cd+1}$, we know from Theorem 1.2 that $d > 16c^3$, which means that all the Diophantine quadruples above are irregular. Since $a_2+b+2\sqrt{a_2b+1} < a_3 + b + 2\sqrt{a_3b+1} \le c$ and $a_1a_2 + 1$ cannot be a square by [10, Theorem 1], $1 \le d_2 = d_-(a_2, b, c) \ne a_1$. Hence, it follows from [3, Corollary 1.4] that $b < d_2$ and from Theorem 1.2 that $c > 16\mu^2b^3 = 16a_1^2b^3$. Since $b > 4000$ by [4, Lemma 3.4], if $b < 4a_1$, then $c > b^5 > 4000b^4$, which contradicts $d_+(a_1, b, c) < d$ and [5, Theorem 1.4]. We thus have $b > 4a_1$. Therefore, $\{a_1, b, c\}$ is a standard triple of the second kind in the sense of [7]. Now, we obtain $c < 10^{2171}$ by [7, Proposition 4] and can conclude $d < 10^{10^{26}}$ as in [7, Section 9]. □

Proof of Corollary 1.7. One proceeds exactly in the same way as in the proof of Corollary 1.6. □

Acknowledgements

The authors thank the referee for careful reading and helpful comments.

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