DIOPHANTINE TRIPLES WITH LARGEST TWO ELEMENTS IN COMMON

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Abstract. In this paper we prove that if $\{a, b, c\}$ is a Diophantine triple with $a < b < c$, then $\{a + 1, b, c\}$ cannot be a Diophantine triple. Moreover, we show that if $\{a_1, b, c\}$ and $\{a_2, b, c\}$ are Diophantine triples with $a_1 < a_2 < b < c < 16b^3$, then $\{a_1, a_2, b, c\}$ is a Diophantine quadruple. In view of these results, we conjecture that if $\{a_1, b, c\}$ and $\{a_2, b, c\}$ are Diophantine triples with $a_1 < a_2 < b < c$, then $\{a_1, a_2, b, c\}$ is a Diophantine quadruple.

1. Introduction

A set of m positive integers $\{a_1, \ldots, a_m\}$ is called *Diophantine m-tuple* if $a_ia_j + 1$ is a perfect square for all i and j with $1 \le i < j \le m$. The second author proved in [9] that there does not exist a Diophantine sextuple and that there exist only finitely many Diophantine quintuples. Recently, it was shown by He, Togbé and Ziegler [17] that there does not exist a Diophantine quintuple, thus confirming a folklore conjecture. On the other hand, the stronger conjecture asserting that all Diophantine quadruples are regular is still open. Here a Diophantine quadruple $\{a, b, c, d\}$ with $a < b < c < d$ is called *regular* if $d = d_+ := a + b + c + 2abc + 2RST$ with $R := \sqrt{ab + 1}$, $S := \sqrt{ac + 1}$, $T := \sqrt{bc + 1}$ (see [1]). For Diophantine quadruples $\{a, b, c, d\}$ with $a < b < c < d$ containing various pairs $\{a, b\}$ or triples $\{a, b, c\}$, such as

- $\{k - 1, k + 1\}$ with $k \ge 2$ an integer [4, 13],
- $\{k, 4k \pm 4\}$ with k a positive integer [12, 14],
- $\{K, A^2K \pm 2A, (A+1)^2K \pm 2(A+1)\}$ with A, K positive integers [6, 15, 16],
- $\{a, b, c\}$ with $c \ge 200b^4$ [7],

it is known that d must be equal to d_+.

In proving each of the results above, the starting point is to transform the conditions that $ad + 1 = X^2$, $bd + 1 = Y^2$, $cd + 1 = Z^2$ for some positive integers X, Y, Z into the system of Pellian equations

$$aZ^2 - cX^2 = a - c,$$
$$bZ^2 - cY^2 = b - c,$$

and the crucial part is where an upper bound for Z is deduced by using Baker's method or hypergeometric method. In any case, the condition that "$ab + 1$ is a perfect square" is not essentially required for the upper bound. This consideration leads us to expect that the following holds.

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Conjecture 1.1. *Suppose that* $\{a_1, b, c\}$ *and* $\{a_2, b, c\}$ *are Diophantine triples with* $a_1 < a_2 < b < c$. *Then,* $\{a_1, a_2, b, c\}$ *is a Diophantine quadruple.*

Conjecture 1.1 together with the result due to He, Togbé and Ziegler [17] implies the following.

Conjecture 1.2. *Suppose that* $\{a_1, b, c, d\}$ *is a Diophantine quadruple with* $a_1 < b < c < d$. *Then,* $\{a_2, b, c, d\}$ *is not a Diophantine quadruple for any integer* a_2 *with* $a_1 \neq a_2 < b$.

Taking the contraposition of Conjecture 1.1, one finds that if $a_1 a_2 + 1$ is not a perfect square for positive integers a_1 and a_2, then at least one of the triples $\{a_1, b, c\}$ and $\{a_2, b, c\}$ is not a Diophantine triple for any integers b and c with $\max\{a_1, a_2\} < b < c$. The first theorem of this paper gives an example of such a pair $\{a_1, a_2\}$.

Theorem 1.3. *Suppose that* $\{a, b, c\}$ *is a Diophantine triple. Then,* $\{a+1, b, c\}$ *is not a Diophantine triple.*

The second theorem of this paper also supports the validity of Conjecture 1.1.

Theorem 1.4. *If* $c < 16b^3$, *then Conjecture 1.1 holds.*

Theorem 1.4 together with [7, Theorem 1.4] and [17, Theorem 1] immediately implies the following.

Corollary 1.5. *If either* $c < 16b^3$ *or* $c \geq 200b^4$, *then Conjecture 1.2 holds.*

The organization of this paper is as follows. The proof of Theorem 1.3 is given in Sections 2 to 4. In Section 2, we express b in terms of a. We show that under a certain assumption (see Assumption 2.2) implying "b and c are minimal", b appears in a sequence (b_ν) $(\nu = 1, 2, \dots)$, and we determine the fundamental solutions to the system of Pellian equations obtained from the conditions on the squareness. We also show the size relations for the indices of sequences which are given as solutions to the system. In Section 3 we give the proof of Theorem 1.3 in the case where $b \geq b_2$ using hypergeometric method developed in [20] (see Theorem 3.2), and in Section 4 we prove Theorem 1.3 in the case where $b = b_1$ using Baker's method on linear forms in two logarithms (see Theorem 4.3). Finally in Section 5 we prove Theorem 1.4 by making use of the properties of regular Diophantine quadruples.

2. Fundamental solutions

Let $\{a, b, c\}$ be a Diophantine triple. Suppose that $\{a+1, b, c\}$ is a Diophantine triple. We may assume that

$$b < c.$$

Let s and t be positive integers satisfying

$$ab + 1 = s^2 \quad \text{and} \quad (a+1)b + 1 = t^2.$$

These equations imply the Pellian equation

$$at^2 - (a+1)s^2 = -1. \tag{1}$$

Since the least positive solution to the Pellian equation $X^2 - a(a+1)Y^2 = 1$ is $(X, Y) = (2a+1, 2)$, following the argument of Nagell [19, Theorem 108a] (see also [8, Lemma 1]), we see that there exists a solution (t_0, s_0) to (1) satisfying

$$0 < s_0 \leq 1, \quad |t_0| \leq 1 \tag{2}$$

such that any positive integer solution (t, s) to (1) can be expressed as

$$t\sqrt{a} + s\sqrt{a+1} = (t_0\sqrt{a} + s_0\sqrt{a+1})(2a + 1 + 2\sqrt{a(a+1)})^\nu \tag{3}$$

for some non-negative integer ν. It is obvious from (2) that $s_0 = 1$ and $t_0 = \pm 1$. Since any positive integer solution (t, s) to (1) with $(t_0, s_0) = (-1, 1)$ is also obtained from $(t_0, s_0) = (1, 1)$, we may take $t_0 = 1$. It follows from (3) that any positive integer solution (t, s) to (1) is given by $s = \sigma_\nu$, where

$$\sigma_0 = 1, \quad \sigma_1 = 4a + 1, \quad \sigma_{\nu+2} = 2(2a + 1)\sigma_{\nu+1} - \sigma_\nu.$$

Put $b_\nu = (\sigma_\nu^2 - 1)/a$. The smallest values of b_ν's are the following:

$$b_0 = 0, \quad b_1 = 16a + 8, \quad b_2 = 256a^3 + 384a^2 + 176a + 24,$$
$$b_3 = 4096a^5 + 10240a^4 + 9472a^3 + 3968a^2 + 736a + 48.$$

We may assume that $b = b_\nu \geq b_1$.

Moreover, there exist positive integers x, y, z such that

$$ac + 1 = x^2, \quad (a + 1)c + 1 = y^2, \quad bc + 1 = z^2,$$

from which we deduce the following system of Pellian equations

$$az^2 - bx^2 = a - b, \tag{4}$$

$$(a + 1)z^2 - by^2 = a + 1 - b. \tag{5}$$

By [8, Lemma 1], we can describe the solutions to (4) and (5) as follows.

Lemma 2.1. *There exist solutions (z_0, x_0) and (z_1, y_1) to (4) and (5), respectively, satisfying*

$$1 \leq x_0 \leq \sqrt{\frac{a(b-a)}{2(s-1)}} < \sqrt{\frac{s+1}{2}},$$

$$1 \leq |z_0| \leq \sqrt{\frac{(s-1)(b-a)}{2a}} < \sqrt{\frac{b\sqrt{b}}{2\sqrt{a}}},$$

$$1 \leq y_1 \leq \sqrt{\frac{(a+1)(b-a-1)}{2(t-1)}} < \sqrt{\frac{t+1}{2}},$$

$$1 \leq |z_1| \leq \sqrt{\frac{(t-1)(b-a-1)}{2(a+1)}} < \sqrt{\frac{b\sqrt{b}}{2\sqrt{a+1}}}$$

such that any positive integer solutions (z, x) and (z, y) to (4) and (5), respectively, can be expressed as

$$z\sqrt{a} + x\sqrt{b} = (z_0 + x_0\sqrt{b})(s + \sqrt{ab})^m,$$
$$z\sqrt{a+1} + y\sqrt{b} = (z_1 + y_1\sqrt{b})(t + \sqrt{(a+1)b})^n$$

for some non-negative integers m and n.

By Lemma 2.1, we may write $z = v_m = w_n$ for some non-negative integers m and n, where

$$v_0 = z_0, \quad v_1 = sz_0 + bx_0, \quad v_{m+2} = 2sv_{m+1} - v_m, \tag{6}$$

$$w_0 = z_1, \quad w_1 = tz_1 + by_1, \quad w_{n+2} = 2tw_{n+1} - w_n. \tag{7}$$

Note that (6) and (7) immediately imply that

$$z_0^2 \equiv z_1^2 \equiv 1 \pmod{b}. \tag{8}$$

In what follows, until the end of the proof of Theorem 1.3, we assume that “b and c are minimal” among the b's and c's for which Theorem 1.3 is not valid, in other words, we put the following.

Assumption 2.2. *At least one of* $\{a, b', b\}$ *and* $\{a+1, b', b\}$ *is not a Diophantine triple for any* b' *with* $0 < b' < b$.

Lemma 2.3. *If the equation* $v_m = w_n$ *has a solution, then both* m *and* n *are even and* $z_0 = z_1 = \varepsilon$, *where* $\varepsilon \in \{\pm 1\}$.

Proof. Suppose first that both m and n are even. By [8, Lemma 3] we have $z_0 = z_1$. Putting $d_0 := (z_0^2 - 1)/b$, which is an integer by (8), we see from Lemma 2.1 that $d_0 < b$. It is clear that $ad_0 + 1 = x_0^2$, $(a+1)d_0 + 1 = y_1^2$ and $bd_0 + 1 = z_0^2$, which means that either $d_0 = 0$ or both $\{a, b, d_0\}$ and $\{a+1, b, d_0\}$ are Diophantine triples. In view of Assumption 2.2, we must have $d_0 = 0$ and $z_0 = \pm 1$.

Suppose second that m is odd and n is even. By [8, Lemma 3] we have $bx_0 - s|z_0| = |z_1|$ and $z_0 z_1 < 0$. Putting $z' := |z_1| = bx_0 - x|z_0|$ and $d_0 := ((z')^2 - 1)/b$, we see from (8) and Lemma 2.1 that d_0 is an integer with $d_0 < b$. Since $ad_0 + 1 = (sx_0 - a|z_0|)^2$, $(a+1)d_0 + 1 = y_1^2$, $bd_0 + 1 = (z')^2$, we deduce from Assumption 2.2 that $d_0 = 0$ and $|z_1| = bx_0 - x|z_0| = 1$. However, the last equality does not hold, since $b \geq b_1 = 16a + 8$ and

$$bx_0 - s|z_0| = \frac{b(b-a) - z_0^2}{bx_0 + s|z_0|} > \frac{2b\sqrt{a} - \sqrt{b} - 2a\sqrt{a}}{2\sqrt{2a(s+1)}} > 5.$$

Therefore, this case does not occur.

Suppose third that m is even and n is odd. By [8, Lemma 3] we have $by_1 - t|z_1| = |z_0|$ and $z_0 z_1 < 0$. Putting $z' := |z_0| = by_1 - t|z_1|$ and $d_0 := ((z')^2 - 1)/b$, one may arrive at a contradiction in the same way as in the previous case.

Suppose finally that both m and n are odd. By [8, Lemma 3] we have $bx_0 - x|z_0| = by_1 - t|z_1|$ and $z_0 z_1 > 0$. Putting $z' := bx_0 - s|z_0| = by_1 - t|z_1|$ and $d_0 := ((z')^2 - 1)/b$, one may again arrive at a contradiction similarly to the previous two cases. □

The following lemma is easily deduced from Lemma 2.3 together with [9, Lemma 3 and its proof].

Lemma 2.4. *If* $v_m = w_n$ *has a solution, then* $n \leq m \leq 2n$.

The previous result can be strengthened as follows.

Lemma 2.5. *If* $v_m = w_n$ *has a solution with* $m \geq 2$ *then* $m > n$.

Proof. If $\varepsilon = 1$, then $v_2 = 2s(b+s) - 1 < 2t(b+t) - 1 = w_2$. If $\varepsilon = -1$, then $v_2 = 2s(b-s) + 1 = 2(s-a)b - 1$ and $w_2 = 2t(b-t) + 1 = 2(t-a-1)b - 1$. Since $t = s+1$ entails $b = t^2 - s^2$ is odd, which is not possible having in view that $b_\nu = (\sigma_\nu^2 - 1)/a$ and $\sigma_\nu \equiv 1 \pmod{4a}$ for any positive ν, we deduce that $v_2 < w_2$.

For $n \geq 3$, we see from $s \leq t - 2$ that

$$\begin{aligned} v_n = 2sv_{n-1} - v_{n-2} < 2sv_{n-1} < 2sw_{n-1} &\leq 2tw_{n-1} - 4w_{n-1} \\ &< 2tw_{n-1} - w_{n-2} = w_n. \end{aligned}$$

By induction, we conclude that if $v_m = w_n$, then $m > n$ for $m \geq 2$. □

3. Proof of Theorem 1.3: The case $b \geq b_2$

Lemma 3.1. *If* $v_m = w_n$ *has a solution with* $m > 0$, *then*

$$m > (a+1)^{-1/2} b^{1/2}.$$

Proof. By (6), (7) and Lemma 2.3 we have

$$\varepsilon a m^2 + 2sm \equiv \varepsilon(a+1)n^2 + 2tn \pmod{16b},$$

that is,

$$\varepsilon\{am^2 - (a+1)n^2\} \equiv 2(tn - sm) \pmod{16b}. \tag{9}$$

Suppose that $m \le (a+1)^{-1/2} b^{1/2}$. Then, since

$$\max\{am^2, (a+1)n^2\} \le (a+1)m^2 \le b,$$
$$\max\{sm, tn\} \le tm \le (a+1)^{-1/2} b^{1/2} \sqrt{(a+1)b+1} < 2b,$$

congruence (9) is in fact an equality. Thus we have

$$\{(a+1)n^2 - am^2\}\{2b + \varepsilon(tn+sm)\} = 2(m^2 - n^2). \tag{10}$$

In the previous proof we have shown that $m \ne n$. Since both m and n are even by Lemma 2.3, we see from Lemma 2.4 that

$$n+2 \le m \le 2n \quad \text{and} \quad |am^2 - (a+1)n^2| \ge 4.$$

It follows from (10) that

$$2|2b + \varepsilon(tn+sm)| \le m^2 - n^2,$$

which yields

$$\begin{aligned} 4b &\le 2(tn+sm) + m^2 - n^2 \le 2\sqrt{(a+1)b+1}(m-2) + 2m\sqrt{ab+1} + \frac{3}{4}m^2 \\ &< \left\{ 2\sqrt{(a+1)b+1}\left(1 - 2\sqrt{\frac{a+1}{b}}\right) + 2\sqrt{ab+1} + \frac{3}{4}\sqrt{\frac{b}{a+1}} \right\} \sqrt{\frac{b}{a+1}}. \end{aligned}$$

Thus, we have

$$2 \le \sqrt{1 + \frac{1}{(a+1)b}}\left(1 - 2\sqrt{\frac{a+1}{b}}\right) + \sqrt{\frac{ab+1}{(a+1)b}} + \frac{3}{8(a+1)},$$

which is a contradiction, since

$$\sqrt{1 + \frac{1}{(a+1)b}}\left(1 - 2\sqrt{\frac{a+1}{b}}\right) < 1 \quad \text{and} \quad \sqrt{\frac{ab+1}{(a+1)b}} + \frac{3}{8(a+1)} < 1.$$

Therefore, we obtain $m > (a+1)^{-1/2} b^{1/2}$. □

Theorem 3.2. *Let a be a positive integer and N a multiple of $a(a+1)$. Assume that $N \ge 4.652a(a+1)^2$. Then, the numbers $\theta_1 = \sqrt{1+(a+1)/N}$ and $\theta_2 = \sqrt{1+a/N}$ satisfy*

$$\max\left\{ \left|\theta_1 - \frac{p_1}{q}\right|, \left|\theta_2 - \frac{p_2}{q}\right| \right\} > \left(1.604 \cdot 10^{28} N\right)^{-1} q^{-\lambda}$$

for all integers p_1, p_2, q with $q > 0$, where

$$\lambda = 1 + \frac{\log(10(a+1)N)}{\log(2.15a^{-1}(a+1)^{-1}N^2)} < 2.$$

Proof. The proof proceeds along the same lines as the one of [5, Theorem 2.2] or [11, Theorem 2.5]. For $0 \le i, j \le 2$ and integers a_0, a_1, a_2, let $p_{ij}(x)$ be the polynomial defined by

$$p_{ij}(x) := \sum_{ij} \binom{k+1/2}{h_j} (1+a_j x)^{k-h_j} x^{h_j} \prod_{l \ne j} \binom{-k_{ij}}{h_l} (a_j - a_l)^{-k_{il}-h_l},$$

where $k_{il} = k + \delta_{il}$ with δ_{il} the Kronecker delta, $\sum_{ij}$ denotes the sum over all non-negative integers h_0, h_1, h_2 satisfying $h_0 + h_1 + h_2 = k_{ij} - 1$, and $\prod_{l \ne j}$ denotes the product from $l = 0$ to $l = 2$ omitting $l = j$ (which is the expression (3.7) in [20] with $\nu = 1/2$). Substituting $x = 1/N$ we have

$$p_{ij}(1/N) = \sum_{ij} \binom{k+1/2}{h_j} C_{ij}^{-1} \prod_{l \ne j} \binom{-k_{ij}}{h_l},$$

where

$$C_{ij} := \frac{N^k}{(N+a_j)^{k-h_j}} \prod_{l\neq j}(a_j-a_l)^{k_{il}+h_l}.$$

We take $a_0 = 0$, $a_1 = a$, $a_2 = a+1$ and $N = a(a+1)N_0$ for some integer N_0. If $j = 0$, then

$$|C_{i0}| = \frac{a^{k_{i1}+h_0+h_1-k}(a+1)^{k_{i2}+h_0+h_2-k}N}{N_0^{k-h_0}} \quad \text{and} \quad a^k(a+1)^kN^kC_{i0}^{-1} \in \mathbb{Z}.$$

If $j = 1$, then

$$|C_{i1}| = \frac{a^{k_{i0}+h_0+h_1-k}N^k}{\{(a+1)N_0+1\}^{k-h_1}} \quad \text{and} \quad a^kN^kC_{i1}^{-1} \in \mathbb{Z}.$$

If $j = 2$, then

$$|C_{i2}| = \frac{(a+1)^{k_{i0}+h_0+h_2-k}N^k}{(aN_0+1)^{k-h_2}} \quad \text{and} \quad (a+1)^kN^kC_{i2}^{-1} \in \mathbb{Z}.$$

Thus we have $\{a(a+1)N\}^kC_{ij}^{-1} \in \mathbb{Z}$ for all i, j. It follows from the proof of [5, Theorem 2.2] that

$$p_{ijk} := 2^{-1}\{4a(a+1)N\}^k\frac{4.09\cdot 10^{13}}{1.6^k}\cdot p_{ij}(1/N) \in \mathbb{Z}$$

and if we put $\theta_0 = 1$, we have

$$|p_{ijk}| < pP^k \quad \text{and} \quad \left|\sum_{j=0}^{2} p_{ijk}\theta_j\right| < lL^{-k},$$

where

$$\begin{aligned}
p &= \frac{4.09\cdot 10^{13}}{2}\left(1+\frac{a}{2N}\right)^{1/2} < 2.073\cdot 10^{13},\\
P &= \frac{32(1+\frac{2a+3}{2N})a(a+1)N}{1.6(2a+1)} < 10(a+1)N,\\
l &= \frac{4.09\cdot 10^{13}}{2}\cdot\frac{27}{64}\left(1-\frac{a+1}{N}\right)^{-1} < 9.667\cdot 10^{12},\\
L &= \frac{1.6}{4a(a+1)N}\cdot\frac{27}{4}\left(1-\frac{a+1}{N}\right)^2 N^3 > \frac{2.15N^2}{a(a+1)}.
\end{aligned}$$

Now, one can deduce Theorem 3.2 from [3, Lemma 3.1], noting that $N \geq 4.652a(a+1)^2$ implies

$$\lambda = 1 + \frac{\log(10(a+1)N)}{\log(2.15a^{-1}(a+1)^{-1}N^2)}$$

and

$$C^{-1} < 4p\cdot\frac{10a(a+1)N}{a}\cdot(2l)^{\lambda-1} < 1.604\cdot 10^{28}(a+1)N.$$

□

Lemma 3.3. (cf. [8, Lemma 12]) *Let $N = a(a+1)b$ and let θ_1, θ_2 be as in Theorem 3.2. Then all positive solutions to the system of Pellian equations* (4) *and* (5) *satisfy*

$$\max\left\{\left|\theta_1 - \frac{(a+1)sx}{a(a+1)z}\right|, \left|\theta_2 - \frac{aty}{a(a+1)z}\right|\right\} < \frac{b}{2a}z^{-2}.$$

Lemma 3.4. *If $m \geq 1$, then $z = v_m > (s+\sqrt{ab})^m$.*

Proof. By (6) and Lemma 2.3, we have

$$v_m = \frac{1}{2\sqrt{a}}\left\{(\varepsilon\sqrt{a}+\sqrt{b})(s+\sqrt{ab})^m + (\varepsilon\sqrt{a}-\sqrt{b})(s-\sqrt{ab})^m\right\}. \tag{11}$$

Note that $b \geq b_1 = 16a+8$. If $\varepsilon = 1$, then

$$v_m > \frac{\sqrt{a}+\sqrt{b}}{2\sqrt{a}}(s+\sqrt{ab})^m\left\{1-\frac{1}{(s+\sqrt{ab})^{2m}}\right\} > (s+\sqrt{ab})^m.$$

If $\varepsilon = -1$, then

$$v_m > (s+\sqrt{ab})^m\left\{\frac{3}{2}-\frac{\sqrt{b}+\sqrt{a}}{2\sqrt{a}}\cdot\frac{1}{(s+\sqrt{ab})^{2m}}\right\} > (s+\sqrt{ab})^m.$$

□

By (7) and Lemma 2.3, we also have

$$\begin{aligned} w_n = \frac{1}{2\sqrt{a+1}}\Big\{&(\varepsilon\sqrt{a+1}+\sqrt{b})(t+\sqrt{(a+1)b})^n \\ &+ (\varepsilon\sqrt{a+1}-\sqrt{b})(t-\sqrt{(a+1)b})^n\Big\}. \end{aligned} \tag{12}$$

Applying standard techniques to $v_m = w_n$ with (11) and (12), we have

$$0 < \Lambda := m\log\alpha - n\log\beta + \log\gamma < \alpha^{1-2m}, \tag{13}$$

where

$$\alpha := s+\sqrt{ab}, \quad \beta = t+\sqrt{(a+1)b} \quad \text{and} \quad \gamma = \frac{\sqrt{a+1}(\sqrt{b}+\varepsilon\sqrt{a})}{\sqrt{a}(\sqrt{b}+\varepsilon\sqrt{a+1})}.$$

Inequality (13) is necessary for the reduction procedure and the proof in the case where $b = b_1 = 16a+8$.

Now we are ready to prove Theorem 1.3 in the case where $b \geq b_2 = 256a^3 + 384a^2 + 176a + 24$.

Proof of Theorem 1.3 *in the case* $b \geq b_2$. Suppose that $b \geq b_2$ and Assumption 2.2 holds. We apply Theorem 3.2 with $N = a(a+1)b$, $p_1 = (a+1)sx$, $p_2 = aty$ and $q = a(a+1)z$. Combining it with Lemma 3.3 shows that

$$z^{2-\lambda} < \frac{b}{2a}\cdot 1.604\cdot 10^{28}a(a+1)b(a(a+1))^{\lambda}.$$

Since

$$2-\lambda = \frac{\log(0.215(a+1)^{-1}b)}{\log(2.15a(a+1)b^2)},$$

we have

$$\log z < \frac{\log(8.02\cdot 10^{27}a^2(a+1)^3b^2)\log(2.15a(a+1)b^2)}{\log(0.215(a+1)^{-1}b)},$$

which together with Lemmas 3.1 and 3.4 implies that

$$(a+1)^{-1}b^{1/2} < \frac{\log(8.02\cdot 10^{27}a^2(a+1)^3b^2)\log(2.15a(a+1)b^2)}{\log(s+\sqrt{ab})\log(0.215(a+1)^{-1}b)}.$$

Since the right-hand side is a decreasing function of b, we see from $b \geq b_2 > 256(a^3+a^2)$ that

$$f(a) := 16a < \frac{\log(5.256\cdot 10^{32}a^8(a+1)^3)\log(1.4091\cdot 10^5a^7(a+1))}{\log(32a^2)\log(55.04a^3(a+1)^{-1})}.$$

Assume that $a \geq 4$. Then the right-hand side of the above equation is less than

$$\frac{\log(1.0266\cdot 10^{33}a^{11})\log(1.7614\cdot 10^5a^8)}{\log(32a^2)\log(44.032a^2)} =: g(a).$$

Since $f(a)$ is an increasing function while $g(a)$ is a decreasing function and $f(4) > 50 > g(4)$, we have $f(a) > g(a)$ for $a \geq 4$, which is a contradiction, Hence, $a \leq 3$.

In the case where $1 \leq a \leq 3$, we repeat the reasoning from the previous paragraph assuming $b \geq b_3 > 4096(a^5+a^4)$ and we readily arrive at a contradiction. Therefore, it remains to consider the pairs $(a,b) = (1,840), (2,3960), (3,10920)$.

A program implementing the variant of Baker-Davenport Lemma [2, Lemma] from [10, Lemma 5] returned the bound $m < 5$, which is not compatible with Lemma 3.1 because $b \geq b_2 > 256a^3$. □

4. Proof of Theorem 1.3: The case $b = b_1$

Lemma 4.1. *If $v_m = w_n$ has a solution with $m \geq 2$, then*

$$(m - 0.001)\log\alpha - n\log\beta < 0.$$

Proof. Since

$$\alpha^3\log\alpha = (s+\sqrt{ab})^3\log(s+\sqrt{ab}) > 2000$$

and $\gamma > 1$, one may deduce from (13) that

$$\Lambda < \alpha^{-3} < 2000^{-1}\log\alpha < 0.001\log\alpha + \log\gamma.$$

This immediately shows the desired inequality. □

Lemma 4.2. *If $v_m = w_n$ has a solution with $m \geq 2$ and $b = b_1 = 16a+8$, then*

$$n > 2(\nu - 0.001)a\log\alpha,$$

where $\nu := m - n$.

Proof. By Lemma 4.1 we have

$$\begin{aligned}\frac{\nu - 0.001}{n} &= \frac{m-0.001}{n} - 1 < \frac{\log\beta}{\log\alpha} - 1 < \frac{\beta-\alpha}{\alpha\log\alpha}\\ &< \frac{2+(\sqrt{a+1}-\sqrt{a})\sqrt{b}}{2\sqrt{ab}\log\alpha} < \frac{4\sqrt{a}+\sqrt{b}}{4a\sqrt{b}\log\alpha}\\ &< \frac{1}{2a\log\alpha},\end{aligned}$$

from which the desired inequality follows. □

Theorem 4.3. ([18, Corollary 2]) *Assume that α_1 and α_2 are real, positive and multiplicatively independent algebraic numbers in a field K of degree D. Set*

$$\Lambda := b_2\log\alpha_2 - b_1\log\alpha_1,$$

where b_1 and b_2 are positive integers. Let A_1 and A_2 be real numbers greater than one such that

$$\log A_i \geq \max\{h(\alpha_i), |\log\alpha_i|/D, 1/D\} \quad (i = 1,2).$$

Set

$$b' := \frac{b_1}{D\log A_2} + \frac{b_2}{D\log A_1}.$$

Then,

$$\log\Lambda > -24.34D^4\left(\max\{\log b' + 0.14, 21/D, 1/2\}\right)^2\log A_1\log A_2.$$

Rewriting Λ as

$$\Lambda = \log(\alpha^\nu\gamma) - n\log\left(\frac{\beta}{\alpha}\right),$$

we apply Theorem 4.3 with

$$b_1 = n,\ b_2 = 1,\ \alpha_1 = \frac{\beta}{\alpha},\ \alpha_2 = \alpha^\nu\gamma,\ D = 4.$$

We have
$$h(\alpha)=\frac{1}{2}\log\alpha \quad \text{and} \quad h(\beta)=\frac{1}{2}\log\beta.$$
Since the conjugates of γ whose absolute values are greater than one are
$$\frac{\sqrt{a+1}(\sqrt{b}+\sqrt{a})}{\sqrt{a}(\sqrt{b}+\sqrt{a+1})},\quad \frac{\sqrt{a+1}(\sqrt{b}-\sqrt{a})}{\sqrt{a}(\sqrt{b}-\sqrt{a+1})},\quad \frac{\sqrt{a+1}(\sqrt{b}+\sqrt{a})}{\sqrt{a}(\sqrt{b}-\sqrt{a+1})},$$
and the leading coefficient of the minimal polynomial of γ is a divisor of $a^2(b-a-1)^2$, we see that
$$h(\gamma)\le\frac{1}{4}\log\left\{a^{1/2}(a+1)^{3/2}(b-a)(\sqrt{b}+\sqrt{a})(\sqrt{b}+\sqrt{a+1})\right\}<\log\alpha.$$
Hence,
$$h(\alpha_1)=h(\beta/\alpha)\le h(\beta)+h(\alpha)=\frac{1}{2}(\log\alpha+\log\beta),$$
$$h(\alpha_2)=h(\alpha^\nu\gamma)\le \nu h(\alpha)+h(\gamma)<\left(\frac{\nu}{2}+1\right)\log\alpha.$$
Moreover, since
$$\gamma\le\frac{\sqrt{a+1}(\sqrt{b}-\sqrt{a})}{\sqrt{a}(\sqrt{b}-\sqrt{a+1})}\le\sqrt{2}\cdot\frac{16a+8+(\sqrt{2}-1)\sqrt{16a+8}-\sqrt{2}}{15a+7}<2,$$
we have
$$\frac{\log\alpha_2}{D}<\frac{\nu\log\alpha+\log 2}{4}<\left(\frac{\nu}{2}+1\right)\log\alpha.$$
Thus, we may take
$$\log A_1=\frac{1}{2}(\log\alpha+\log\beta),\quad \log A_2=\left(\frac{\nu}{2}+1\right)\log\alpha,$$
which together with $n\le m-2$ yields
$$b'=\frac{n}{2(\nu+2)\log\alpha}+\frac{1}{2(\log\alpha+\log\beta)}<\frac{m}{2(\nu+2)\log\alpha}.$$
Since
$$\beta=\frac{t+\sqrt{(a+1)b}}{s+\sqrt{ab}}\cdot\alpha<1.41\alpha$$
and
$$\log\alpha+\log\beta<\log(1.41\alpha^2)<2.15\log\alpha,$$
it follows from (13) and Theorem 4.3 that
$$\frac{m-0.5}{2(\nu+2)\log\alpha}<52.331\left(\max\left\{\log\left(\frac{m}{2(\nu+2)\log\alpha}\right),5.25\right\}\right)^2. \tag{14}$$
If $\log(m/(2(\nu+2)\log\alpha))\le 5.25$, then
$$m<382(\nu+2)\log\alpha.$$
If $\log(m/(2(\nu+2)\log\alpha))>5.25$, then inequality (14) implies that
$$m<6960.2(\nu+2)\log\alpha. \tag{15}$$
Thus, inequality (15) holds in any case. Combining Lemma 4.2 with (15), we obtain
$$2(\nu-0.001)a<6960.2(\nu+2),$$
which yields
$$a<6964$$
It therefore remains to prove the theorem for $a<6964$.

Two steps of the reduction process ended with the bound $m<5$. From Lemmas 2.4 and 2.5 one deduces $m=4$, $n=2$, and it is now an easy task to explicitly compute the relevant values v_m, w_n and see they are different.

5. Proof of Theorem 1.4

Put

$$d_i := a_i + b + c + 2a_ibc - 2r_is_iu \quad \text{for} \quad i \in \{1,2\},$$

where

$$r_i := \sqrt{a_ib+1}, \quad s_i := \sqrt{a_ic+1}, \quad u := \sqrt{bc+1}.$$

It is well known that $0 \le d_i < c$ and it holds

$$(b+c-a_i-d_i)^2 = 4(a_id_i+1)(bc+1) \tag{16}$$

for $i \in \{1,2\}$. Moreover, if $d_i > 0$, then $\{a_i, d_i, b, c\}$ is a Diophantine quadruple, in particular, $t_i := \sqrt{a_id_i+1}$ is an integer.

Noting that

$$c = 4a_id_ib + \lambda_i \max\{d_i, b\}, \tag{17}$$

with λ_i a rational number satisfying $1 < \lambda_i < 4$, we have

$$4(a_1d_1 - a_2d_2)b = \lambda_2 \max\{d_2, b\} - \lambda_1 \max\{d_1, b\}. \tag{18}$$

Thus we obtain

$$|a_1d_1 - a_2d_2| < \frac{\max\{d_1, d_2, b\}}{b}. \tag{19}$$

If $\max\{d_1, d_2, b\} = b$, then from (18) we get $4|a_1d_1 - a_2d_2| = |\lambda_2 - \lambda_1| < 3$, which implies that $|a_1d_1 - a_2d_2| < 1$ and $a_1d_1 = a_2d_2$.

If $\max\{d_1, d_2, b\} = d_1$, suppose that $a_1d_1 \ne a_2d_2$. Then we have

$$|a_1d_1 - a_2d_2| = |t_1^2 - t_2^2| \ge |t_1^2 - (t_1-1)^2| = 2t_1 - 1,$$

which together with (19) shows that

$$2t_1 < \frac{d_1}{b} + 1.$$

Squaring both sides of this inequality yields

$$d_1^2 - 2b(2a_1b-1)d_1 - 3b^2 > 0,$$

which means that $d_1 > 2b(2a_1b-1)$. This in turn implies that $t_1 > 2a_1b - 1$. As t_1 is coprime with a_1, for $a_1 \ge 2$ one has $t_1 \ge 2a_1b+1$. Assuming $a_1 = 1$ and $t_1 = 2b$, one obtains $d_1 = t_1^2 - 1 = 4b^2 - 1$ and, by (17),

$$c > 4b(4b^2-1) + 4b^2 - 1 > 16b^3.$$

Thus, $t_1 \ge 2a_1b + 1$ holds in any case. Then from (17) it follows that

$$c > 4a_1d_1b \ge 16a_1b^2(a_1b+1) > 16b^3,$$

which contradicts the hypothesis $c < 16b^3$. Hence, we get $a_1d_1 = a_2d_2$.

If $\max\{d_1, d_2, b\} = d_2$, then $a_1d_1 < a_2d_2$ and in the same way as in the previous case we obtain $d_2 > 2b(2a_2b-1)$ and $c > 16b^3$, a contradiction. Hence, this case cannot occur.

Therefore, we have seen that $a_1d_1 = a_2d_2$.

Equation (16) together with $a_1d_1 = a_2d_2$ implies that

$$(b+c-a_1-d_1)^2 = (b+c-a_2-d_2)^2.$$

Since $a_i < b$ and $d_i < c$, we obtain $a_1 + d_1 = a_2 + d_2$, which combined with $a_1d_1 = a_2d_2$ yields $d_1 = a_2$ and $d_2 = a_1$. This implies that $\{a_1, a_2, b, c\}$ is a Diophantine quadruple.

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